

# Targets, local weak $\sigma$ -Gibbs measures and a generalized Bowen dimension formula

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## Abstract

For a dynamical system, we study the set of points  $\mathcal{W}$  whose orbit approximates any chosen point at certain specified rates. Our basic setting is that of left shift acting on topological Markov chains endowed with a local weak Gibbs measure. Our rates of recurrence are so fast that the corresponding set  $\mathcal{W}$  has measure zero, but we obtain a generalized Bowen formula for Carathéodory dimension. For the case of Markov transformations with countable partition and big image (BI) property a Bowen-type formula is obtained for the Hausdorff dimension of those exceptional sets. In particular, we apply our general results to Gauss and Luroth maps, a not Bernoulli modification of Gauss map and some inner functions. Since we only require the existence of weak Gibbs measures we can deal with non Hölder potentials and we can also consider intermittent systems as the Manneville-Pomeau map.

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## 1 Introduction

Let  $(X, d)$  be a locally complete separable metric space endowed with a finite measure  $\mu$  over the Borel sets. The classical recurrence theorem of Poincaré says that if a measurable transformation  $T : X \rightarrow X$  preserves the measure  $\mu$ , then  $\mu$ -almost every point of  $X$  is recurrent, in the sense that

$$\liminf_{n \rightarrow \infty} d(T^n(x), x) = 0.$$

Here  $T^n$  denotes the  $n$ -th fold composition  $T^n = T \circ T \circ \dots \circ T$ .

It is natural to ask if the orbit  $\{T^n(x)\}$  of the point  $x$  returns regularly not only to every neighborhood of  $x$  itself as Poincaré's theorem asserts, but whether it also visits every neighborhood of a previously chosen point  $y \in X$ . Under the additional hypothesis of ergodicity it is easy to check (see e.g. [14]) that for any  $y \in X$ , we have that

$$\liminf_{n \rightarrow \infty} d(T^n(x), y) = 0, \quad \text{for } \mu\text{-almost all } x \in X. \quad (1)$$

Recall that  $\mu$  is ergodic if the only  $T$ -invariants sets (up to sets of  $\mu$ -measure zero) are trivial, i.e. they have zero  $\mu$ -measure or their complement have zero  $\mu$ -measure.

In this paper we will consider quantitative versions of (1). We are interested in studying the size of the set of points of  $X$  such that  $\liminf_{n \rightarrow \infty} d(T^n(x), y)/r_n = 0$ , where  $\{r_n\}$  is a given sequence of positive numbers and  $y$  is a previously chosen point in  $X$ . So, we study the size of the set

$$\mathcal{W}_T(X, r_n, y) = \{x \in X : d(T^n(x), y) < r_n \text{ for infinitely many } n\}. \quad (2)$$

If  $d(T^k(x), y) < r_k$ , then the orbit of  $x$  “hits” at time  $k$  the target-ball of center  $y$  and radius  $r_k$ . We will refer to  $\mathcal{W}_T(X, r_n, y)$  as the target-ball set.

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### *Aim of this work*

If  $\mu$  is ergodic and the sequence  $\{r_n\}$  is bounded below by some positive number, it follows from (1) that the set  $\mathcal{W}_T(X, r_n, y)$  has full  $\mu$ -measure. In [15], under so called eventual quasi-independence property, we proved that if  $\sum_n \mu(B(y, r_n)) = \infty$  then the set  $\mathcal{W}_T(X, r_n, y)$  has positive  $\mu$ -measure. Now, we would like to study in more detail the case  $\mathcal{W}_T(X, r_n, y)$  being an exceptional set, i.e with zero  $\mu$ -measure. Notice that by the direct part of Borel-Cantelli lemma we know that if  $\sum_n \mu(B(y, r_n)) < \infty$ , then  $\mathcal{W}_T(X, r_n, y)$  has zero  $\mu$ -measure.

In this paper we consider the generic symbolic setting consisting of the left-shift acting on a topological Markov chain endowed with a local weak Gibbs measure, and we obtain a generalized Bowen formula for a Caratheodory dimension of the target-ball set.

This setting is particularly interesting due to the fact that hyperbolic systems are modeled by countable Markov chains (via Markov partitions). And so one can expect this study to provide a useful framework when considering similar questions in such systems. In particular, when the transformation is conformal enough, the Hausdorff dimension for the target-ball set would follow from the corresponding results for topological Markov chains. Following this approach, we also consider in this paper the more concrete setting of Markov transformations with the ACIP measure (absolutely continuous with respect to Lebesgue measure invariant probability). We use the Carathéodory dimension results in the symbolic setting to establish Hausdorff dimension results for Markov transformations.

Since in our symbolic setting we just ask for *weak* Gibbs measures we are able to deal with hyperbolic system with less regularity, and in particular we get Hausdorff dimension results for the target-ball set in the case of intermittent systems. In a forthcoming work [34] we consider the non-additive thermodynamic formalisms for countable Markov shifts and this allows the study of more general target sets.

### *Some previous results*

If the set  $T^{-n}(B(y, r_n))$  is a countable union  $\mathcal{W}_n$  of pairwise disjoint balls, then the points in the target ball are those  $x$  such that  $d(x, c_i) \leq R_i$  for infinitely many  $i$  with  $c_i$  the center and  $R_i$  the radius of a ball in  $\mathcal{W} := \cup_n \mathcal{W}_n$ . These points are called *well approximable points* by the system  $\mathcal{W}$  in diophantine approximation theory. Having a good distribution by size of the balls of the system one gets Hausdorff dimension results for the well approximable set. This notion was developed by Baker and Schmidt, refining some ideas of Besicovitch [7], when they introduced in [4] the concept of *regular system of intervals*. In that work they proved a generalization of the Jarnik-Besicovitch theorem considering approximation by algebraic numbers. In [32] we extended the idea of regular system to *well-distributed system of balls* to obtain Hausdorff dimension results of the same kind. Similar diophantine approximation results were obtained using a different approach in [53]. We would like to remark that our Cantor-like constructions in this paper (also in [33]) are inspired, in a broad sense, by the idea of regular systems. See section 1.3 of this introduction for a brief description.

The Hausdorff dimension of the target-ball set for different kind of transformations has been studied for several authors. In [19] and [20] Hill and Velani studied the case of  $T$  being an expanding rational map of the Riemann sphere acting on its Julia set. Urbanski in [52] considered the case of conformal *countable* iterated function systems. In both cases the value of the Hausdorff dimension of the target-ball set was expressed as the zero of a Bowen formula. In section 1.1 we explain the symbolic counterpart of these results.

We considered in [16] the general framework of expanding systems in metric spaces getting lower and upper bound estimates on the dimension of target-ball and target-block sets. In this setting we have a sequence of partitions  $\{\mathcal{P}_n\}$  of the space with  $\mathcal{P}_{n+1}$  finer than  $\mathcal{P}_n$ , and so the target-block set is defined by changing the role of the balls centered at  $y$  by elements of the partitions containing the point  $y$ . In particular we proved Hausdorff dimension estimates for the target-ball/block sets for the Gauss map. Afterwards, Wang and Zhang got in [56] the Hausdorff dimension for the block-target set for the Gauss map. In [24] Li, Wang, Wu and Xu also studied the Gauss map and obtained the Hausdorff dimension for a more general version of the target-ball set. They consider in (2) a

sequence  $\{y_n\}$  instead of  $y$  and  $\phi(n, x) = e^{-(f(x)+\dots+f(T^{n-1}(x)))}$  with  $f$  a positive continuous function instead of  $r_n$ . Previously, in [55] Wang and Wu got the Hausdorff dimension for the case  $y = 0$ .

More recently, the dimension size of the recurrent version of the set (2), i.e with  $y = x$ , has been studied in [51] by Tan and Wang for  $\beta$ -expansions, and by Seuret and Wang in [49] for conformal iterated function system.

Before we describe in detail our results, we would like to summarize the main novelties contained in the paper:

1. By considering the target-ball set in the setting of topological Markov chains endowed with a local weak Gibbs measure, we found a common substrate implicit in the examples of hyperbolic systems mentioned above.
2. Using the symbolic results, we have got the Hausdorff dimension of the target-ball set for Markov transformations having the BIP property.
3. By considering *weak* Gibbs measures in the symbolic setting we computed the Hausdorff dimension for the target-ball set of some intermittent systems.
4. A generalized Hungerford lemma for Cantor-like sets on topological Markov chains is established.

## 1.1 Target problem for Markov chains

Our dynamic setting can be described as follows:

*Space, distance and transformation*

Given a countable alphabet  $\mathcal{I}$  we denote by  $\Sigma^{\mathcal{I}}$  the space of all infinite words formed with this alphabet. In  $\Sigma^{\mathcal{I}}$  we define the distance

$$d((i_k), (j_k)) = 2^{-m} \quad \text{with} \quad m = \min\{n \in \mathbb{N} : i_n \neq j_n\},$$

where  $\inf \emptyset := \infty$ . The left shift  $\sigma : \Sigma^{\mathcal{I}} \rightarrow \Sigma^{\mathcal{I}}$  is the continuous map defined by  $\sigma(i_0, i_1, i_2, \dots) = (i_1, i_2, \dots)$ . Moreover, given a  $\mathcal{I} \times \mathcal{I}$  transition matrix  $A = (a_{i,j})$  with entries 0 and 1, we consider the  $\sigma$ -invariant subset  $\Sigma_A^{\mathcal{I}} \subset \Sigma^{\mathcal{I}}$  defined by

$$\Sigma_A^{\mathcal{I}} = \{(i_0, i_1, \dots) \in \Sigma^{\mathcal{I}} : a_{i_k, i_{k+1}} = 1 \text{ for all } k = 0, 1, \dots\}.$$

The triple  $(\Sigma_A^{\mathcal{I}}, d, \sigma)$  is called a (one-sided) *topological Markov chain*. Along this paper we will assume that our topological Markov chain is topologically mixing (see definition in section 2.1).

The *n-cylinders* in  $\Sigma_A^{\mathcal{I}}$  (which generate the topology) are the sets

$$C_{i_0 i_1 \dots i_n} = \{(k_0, k_1, \dots) \in \Sigma_A^{\mathcal{I}} : k_s = i_s \text{ for all } s = 0, 1, \dots, n\},$$

and for  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathcal{I}}$ , we denote the *n-cylinder*  $C_{w_0 w_1 \dots w_n}$  by  $C(n, w)$ .

*Measure*

We endow our topological Markov chain  $\Sigma_A^{\mathcal{I}}$  with a  $\sigma$ -invariant local weak Gibbs measure  $\hat{\mu}$  associated to some potential  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$ , (see definition 3.1). Considering weak Gibbs measures (instead of Gibbs) will allow us to deal with non Hölder potentials. The “local” term just means that we add a constant depending on the first symbol in the usual definition of the weak Gibbs measure of a cylinder. This will be useful for working with Ruelle-Perron-Frobenius (RPF) measures in the infinite countable case with BI property.

*Target-ball set*

Given a point  $w \in \Sigma_A^{\mathcal{I}}$ , a *N-cylinder*  $\hat{P}$ , and a sequence  $\{\ell_n\} \subset \mathbb{N}$  our target-ball set is

$$W_{\sigma}(\hat{P}, 2^{-(\ell_n+1)}, w) = \{z \in \hat{P} : \sigma^k(z) \in C(\ell_k, w) \text{ for infinitely many } k\}.$$

If  $\sigma^k(z) \in C(\ell_k, w)$ , then  $z_{k+j} = w_j$  for  $0 \leq j \leq \ell_k$ , and therefore  $d(\sigma^k(z), w) \leq 2^{-(\ell_k+1)}$ . For simplicity we write  $W_\sigma(\hat{P}, \ell_n, w)$  instead of  $W_\sigma(\hat{P}, 2^{-(\ell_n+1)}, w)$

Theorem A for finite alphabet and theorem B for infinite countable alphabet, below, give Bowen formulas for the  $\hat{\mu}$ -dimension (defined by Carathéodory's construction, see section 2) of these target-ball sets.

Our approach to study these sets is based on the thermodynamic formalism for countable Markov shifts. Thermodynamic formalism goes back to the classical work of Ruelle, Sinai and Bowen ([40], [50], [8]) in which they translate the ideas of Gibbsian statistical mechanics to the field of dynamical systems. For infinite countable alphabet, the thermodynamic formalism has been studied by many authors, (see e.g [18], [30], [31], [43], [57]). In this paper we follow Sarig's approach, see [47], [48], and we use the Gurevich pressure and Sarig's results on the existence of equilibrium and Gibbs measures based on his study of Ruelle's operator.

For the sake of clarity of exposition next we split the discussion into finite and infinite countable alphabet. We will assume that

$$s := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \hat{\mu}(C(\ell_n, w)) < \infty.$$

If  $\hat{\mu}$  is weak Gibbs but not Gibbs, then we also require  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ . If  $s = \infty$  then the target-ball set has zero  $\hat{\mu}$ -dimension; if  $\lim_{n \rightarrow \infty} \ell_n/n = \infty$  and  $\hat{\mu}$  is ergodic with  $\phi \in L^1(\hat{\mu})$  and  $\phi$  with enough regularity (see section 5.2), then  $s = \infty$  for  $w$   $\hat{\mu}$ -a.e. We recall that  $\phi$  denotes the potential of our measure  $\hat{\mu}$ .

### 1.1.1 Finite alphabet

We have the following formula for the  $\hat{\mu}$ -dimension of the target-ball set (see theorem 5.3).

**Theorem A.** *If there exists a mixing weak  $\sigma$ -Gibbs measure with continuous potential  $t\phi$  for all  $0 < t \leq 1$ , then*

$$\text{Dim}_{\hat{\mu}}(W_\sigma(\hat{P}, \ell_n, w)) = T,$$

with  $T$  the unique root of the equation

$$P_{\text{top}}(t\phi) - P_{\text{top}}(\phi)t = st. \quad (3)$$

Here  $P_{\text{top}}(\cdot)$  denote the classical topological pressure.

As stated we only need mixing weak  $\sigma$ -Gibbs measures for the potentials  $t\phi$ . But, if  $\phi$  satisfies Walter's condition (see section 3), then from classical results of Bowen and Ruelle we know that there is an exact (whence ergodic and strongly mixing)  $\sigma$ -Gibbs measure with potential  $t\phi$ , for all  $0 < t \leq 1$ . Notice that for any measurable set  $E \subset \Sigma_A^{\mathbb{Z}}$  we have that  $\text{Dim}_{\hat{\mu}}(E) \leq 1$ , and if  $\hat{\mu}(E) > 0$ , then  $\text{Dim}_{\hat{\mu}}(E) = 1$ .

We can rewrite (3) as a Bowen equation, more precisely as

$$P_{\text{top}}(t\psi) = 0 \quad \text{with} \quad \psi = \phi - P_{\text{top}}(\phi) - s.$$

Bowen, in his study of quasi-circles [9], first described what we now call Bowen's formula, the Hausdorff dimension of a set as the solution of a pressure equation. Bowen's result was extended by Ruelle [42] to the repeller  $J$  of a  $C^{1+\varepsilon}$  map  $f$  which is conformal and topologically mixing on  $J$ . The Hausdorff dimension of the conformal repeller  $J$  is the unique root  $t_0$  of the equation  $P_{\text{top}}(-t\phi) = 0$  with  $\phi : J \rightarrow \mathbb{R}$  defined by  $\phi(x) = \log \|D_x f\|$ , and moreover the  $t_0$ -dimensional Hausdorff measure of  $J$  is positive and finite and equivalent to the Gibbs measure with potential  $-t_0\phi$ .

In [20] Hill and Velani studied the target-ball set for  $f$  an expanding rational map of the Riemann sphere acting on its Julia set  $J$ . They proved that the Hausdorff dimension of  $W_f(J, e^{-n\tau}, y)$  was the unique root of the Bowen equation  $P_{top}(-t \log |f'|) = t\tau$ .

The symbolic version of Hill and Velani result corresponds to considering, in theorem A, the Markov chain associated (via Markov partition) to the Julia set  $J$ , and  $\hat{\mu}$  as the  $\sigma$ -Gibbs measure for the potential  $\phi = -t_0 \log |f' \circ \pi|$ , with  $t_0$  is the Hausdorff dimension of  $J$ . (Here  $\pi$  denotes the standard projection from the symbolic model to the Julia set). From the thermodynamic formalism we know that the Hölder continuity of the potential  $\phi$  implies the existence of  $\sigma$ -Gibbs measures with potentials  $t\phi$ . To derive a Hausdorff dimension formula for  $W_f(J, e^{-n\tau}, y)$ , via our symbolic modeling and Carathéodory dimension, conformality is, of course, crucial, but we will not pursue this matter here.

We would like to remark that some recent results show the existence of weak  $\sigma$ -Gibbs measures for non-Hölder potentials, see e.g [17], [22], [58], [59], [23]. Consideration of weak  $\sigma$ -Gibbs measures will allow us to deal with hyperbolic systems with less regularity. As an example, in section 5.2.3, we consider some non-Hölder potentials studied by Hu in [22].

### 1.1.2 Infinite countable alphabet

Sarig proved in [46] that under the *big images and preimages* (BIP) property, the RPF measure (coming from his generalization of Ruelle's Perron Frobenius theorem) is a  $\sigma$ -Gibbs measure for any potential with enough regularity. A closely related result was simultaneously obtained by Mauldin and Urbanski in [30]. We recall that the BIP property is the following condition on the transition matrix of  $\Sigma_A^{\mathcal{I}}$ : there exist a finite set of symbols  $\mathcal{I}_0 \subset \mathcal{I}$  such that for each symbol  $i \in \mathcal{I}$  there exist  $k, \ell \in \mathcal{I}_0$  such that  $C_{kil} \neq \emptyset$ .

The existence of local weak  $\sigma$ -Gibbs measures for potentials  $t\phi$  (with  $0 < t \leq 1$ ) is an important tool in our estimation from below of the  $\hat{\mu}$ -dimension of the target-ball set. We use these measures to construct Cantor-like sets inside  $W_\sigma(\hat{P}, \ell_n, w)$  with large  $\hat{\mu}$ -dimension. Hence, a more precise result is obtained if we assume BIP property. Getting the upper bound of the  $\hat{\mu}$ -dimension of the target-ball set is easier (see proposition 5.1).

In theorem 5.4 we state our  $\hat{\mu}$ -dimension result under the weaker condition *Big images* (BI) property. This condition is equivalent to the condition  $\inf\{\hat{\mu}(\sigma(C_i)) : i \in \mathcal{I}\} > 0$ . Our more general result for the lower bound of  $\text{Dim}_{\hat{\mu}} W_\sigma(\hat{P}, \ell_n, w)$  is corollary 5.1.

If the BIP property holds, and  $\hat{\mu}$  is a  $\sigma$ -Gibbs measure with potential  $\phi$  such that  $\sum_{n=1}^{\infty} V_n(\phi) < \infty$  with

$$V_n(\phi) := \sup\{|\phi(w) - \phi(w')| : w = (i_0, i_1, \dots), w' = (j_0, j_1, \dots) \in \Sigma_A^{\mathcal{I}}, i_k = j_k, 0 \leq k \leq n-1\},$$

and  $\sup \phi < \infty$ , then we have the following (see theorem 5.6) :

**Theorem B.** *If there exists  $0 < t_1 \leq 1$  such that*

$$\infty > P_G(t_1\phi) - P_G(\phi)t_1 > st_1 \quad \text{and} \quad -\sum_{i \in \mathcal{I}} \hat{\mu}(C_i)^{t_1} \log \hat{\mu}(C_i) < \infty,$$

*then*

$$\begin{aligned} \text{Dim}_{\hat{\mu}}(W_\sigma(\hat{P}, \ell_n, w)) &= \sup\{t \geq t_1 : P_G(t\phi) - P_G(\phi)t > st\} \\ &= \inf\{t > 0 : P_G(t\phi) - P_G(\phi)t < st\}. \end{aligned}$$

*And if moreover  $\phi$  is weakly Hölder continuous, then*

$$\text{Dim}_{\hat{\mu}}(W_\sigma(\hat{P}, \ell_n, w)) = T$$

*with  $P_G(T\phi) - P_G(\phi)T = sT$ . Here  $P_G(\cdot)$  denotes the Gurevich pressure*

The function  $t \rightarrow P_G(t\phi) - P_G(\phi)t = P_G(t[\phi - P_G(\phi)])$  is not necessarily continuous, this was first pointed out by Mauldin and Urbanski in [29] (for their pressure); see also [31]. Hence the 'sup' above is not necessarily a maximum.

In [52] Urbanski studied the target-ball problem for the limit set of a conformal *countable* iterated function system, and got a result like Bowen's for the Hausdorff dimension. We recall that an iterated function system is a collection of injective contractions (all with the same contractive constant). The symbolic version of his result corresponds to considering, in theorem B, Markov chains with the *Bernoulli* property (i.e. all the entries in the transition matrix are 1), and a potential with strong regularity properties. Notice that BIP property is weaker than Bernoulli property.

In section 5.1.2 we give an alternative approach to get a lower estimate of the  $\hat{\mu}$ -dimension of  $W_\sigma(\hat{P}, \ell_n, w)$ , in this case we do not require the existence of new measures, but we ask for the initial measure  $\hat{\mu}$  to be mixing. We get the lower bound

$$\text{Dim}_{\hat{\mu}}(W_\sigma(\hat{P}, \ell_n, w)) \geq \frac{P_G(\phi) - \int \phi d\hat{\mu}}{P_G(\phi) - \int \phi d\hat{\mu} + s}.$$

We can think of this value as a lower bound for the root  $T$  in theorems A and B. We would like to mention that in [16] we obtained a related bound for the target-ball set in the general setting of expanding maps in metric spaces.

For example, we have the following result that corresponds to the symbolic version of the Gauss map (see remark 5.11).

**Theorem C.** *Let  $\hat{\mu}$  be the RPF probability for the potential  $\phi : \Sigma^{\mathbb{N}} \rightarrow \mathbb{R}$  defined by*

$$\phi(w) = 2 \log |\pi(w)| \quad \text{with} \quad \pi(w) = \lim_{j \rightarrow \infty} \frac{1}{w_0 + \frac{1}{w_1 + \frac{1}{\ddots + w_j}}}$$

*Then*

$$\text{Dim}_{\hat{\mu}}(W_\sigma(\hat{P}, \ell_n, w)) = T \geq \frac{h_{\hat{\mu}}}{h_{\hat{\mu}} + s} = \frac{\pi^2}{\pi^2 + (6 \log 2)s}$$

*where  $1/2 < T \leq 1$  is the unique solution of  $P_G(t\phi) = st$ . Here  $h_{\hat{\mu}}$  denotes the entropy of  $\hat{\mu}$ .*

## 1.2 Markov transformations and Intermittent systems

Next we turn to the target-ball problem for  $f$  a Markov transformation. In this case our space is the interval  $[0, 1]$ , distance is euclidean, and the  $f$ -invariant probability is the ACIP measure  $\mu$ . Our objective is to get estimates for the Hausdorff dimension of target sets for  $f$ . Recall that any Markov transformation  $f$  has a symbolic representation (via the Markov partition  $\mathcal{P}_0$ ). Our approach is to use this representation and the Carathéodory dimension results described in the previous section.

Let  $(\Sigma_A^{\mathcal{I}}, \sigma)$  be the symbolic representation of  $f$ , let  $\pi$  denote the standard projection from  $\Sigma_A^{\mathcal{I}}$  to the interval  $[0, 1]$ , and  $\hat{\mu} = \mu \circ \pi$  be the corresponding measure. We will assume that the BI property holds, then  $\hat{\mu}$  is a local  $\sigma$ -Gibbs measure with potential  $-\log |f' \circ \pi|$ ; see proposition 5.4.

The collection  $\{\mathcal{P}_n := \bigvee_{j=0}^n f^{-j}(\mathcal{P}_0)\}$  of partitions of the interval  $[0, 1]$  give us a *grid* of  $[0, 1]$ ; we use  $P(n, z)$  to denote the  $n$ -block in  $\mathcal{P}_n$  which contains the point  $z$ . There is a direct correspondence between the target-ball set in  $(\Sigma_A^{\mathcal{I}}, \sigma)$  and the target-*block* set for  $f$ , i.e the set of points in  $[0, 1]$  whose orbit by  $f$  hits at time  $k$  (for infinitely many  $k$ ) the  $\ell_k$ -block of the partition  $\mathcal{P}_{\ell_k}$  which contains the point  $x := \pi(w)$ . In particular for  $f(x) = 10x \pmod{1}$  or  $f$  the Gauss map, the study of the size of the target-block set is, in fact, a very classical problem in diophantine approximation, the study of exceptional sets with prescribed string of digits in his decimal or continued fraction expansions. (See section 6.3 for results on continued fraction and Luroth expansions).

Our dimension results for the target-ball set for  $f$  come from consider an appropriate target-block set inside. The  $\hat{\mu}$ -dimension results in  $(\Sigma_A^{\mathcal{I}}, \sigma)$  will translate into *grid*-dimension results in the interval  $[0, 1]$  for  $f$ . The grid-dimension is defined as the Hausdorff dimension but in the coverings only



intervals of the grid intervene. Grid and Hausdorff dimensions coincide for a set  $A$  (see proposition 2.1) if for all  $\gamma > 0$  and for all  $z \in A$

$$\frac{\text{diam}(P(n, z))}{\text{diam}(P(n-1, z))^{1+\gamma}} \geq C > 0 \quad (4)$$

For Markov transformations with finite alphabet, condition (4) holds (with  $\gamma = 0$ ) for *all* points in  $[0, 1]$ , and we have equality between both dimensions, but this is not true for countable infinite alphabet.

For our Hausdorff dimension results we require that condition (4) holds on the Cantor-like sets constructed to get the lower bounds of the dimension. We get this property by requiring, in the symbolic framework, for a similar condition involving the RPF measure with potential  $-t \log |f' \circ \pi|$  (in the case this potential is positive recurrent) instead of the diameter. Essentially, we use that there is a fixed proportion (in measure) of *good* points (see definitions 3.2 and 3.3) in a collection  $\mathcal{D} = \{C(0, \sigma^{p_i}(w))\}$  of 0-cylinders with  $\{p_i\}$  an increasing sequence in  $\mathbb{N}$  and  $w$  the target-center. Good points come from uniform convergence in the Birkhoff's ergodic theorem for the potential of the measure.

If the collection  $\mathcal{D}$  is finite, then we obtain this fixed proportion of *good* points in  $\mathcal{D}$  by using that the RPF measure is mixing. Therefore the ergodicity of the ACIP and the recurrence theorem of Poincaré (see (1)) give us the desire proportion of good points for  $w$   $\lambda$ -a.e.

However, it is interesting to understand the mixing properties behind this proportional distribution of good points in any (countable) collection of 0-cylinders and for any local *weak*  $\sigma$ -Gibbs measure (and not only for a local  $\sigma$ -Gibbs measure with potential  $-t \log |f' \circ \pi|$ ). Surely in forthcoming situations (more general than Markov transformations) it will be useful to go from grid to Hausdorff dimension. With this purpose, in section 3.2 we define a new mixing property which implies having a fixed proportion of good points in a collection of 0-cylinder (see theorem 3.1). This property is a generalization of a mixing condition already used in [15]. In particular, if the potential  $-t \log |f' \circ \pi|$  is positive recurrent and BIP property holds, then the RPF measure is a local  $\sigma$ -Gibbs measure which satisfies this mixing property in the collection of *all* 0-cylinders. In fact, the RPF measure satisfies a stronger mixing property which is called exponentially continued fraction mixing.

Section 6 contains our results on the Hausdorff dimension of targets sets for Markov transformations with BI property. A very precise result for Markov transformation with BIP property is stated in section 6.2. Recall that BIP property is weaker than Bernoulli property. In particular, in section 6.3 we consider some concrete examples such as: the Gauss transformation, a not Bernoulli modification of the Gauss transformation, the Luroth map and some inner (analytic) functions.

For example, we consider the following (not Bernoulli) modification of the Gauss map  $\phi(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$

$$f(x) = \begin{cases} \phi(x), & \text{if } \frac{1}{2} < x \leq 1, \\ \left(1 - \frac{1}{\lfloor \frac{1}{x} \rfloor}\right) \phi(x) + \frac{1}{\lfloor \frac{1}{x} \rfloor}, & \text{if } 0 < x \leq \frac{1}{2}. \end{cases}$$

The initial partition for  $f$  is, as in the Gauss map,  $\mathcal{P}_0 = \{P_i^0 := (1/(i+1), 1/i) : i \in \mathbb{N} \setminus \{0\}\}$ , but in this case  $f(P_1^0) = (0, 1)$  and  $f(P_i^0) = (1/i, 1)$  for  $i \geq 2$ .

Let  $\{r_n\}$  be a sequence of radii such that  $u := -\lim_{n \rightarrow \infty} \frac{1}{n} \log r_n < \infty$ , then (see theorem 6.5)

**Theorem D.** *Let  $1/2 < T \leq 1$  be the unique solution of  $P_G(-t \log |f' \circ \pi|) = tu$ , then for  $\lambda$ -a.e  $x \in (0, 1)$*

$$\text{Dim} \left\{ y \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = 0 \right\} = T \geq \frac{\int \log |f'| d\mu}{\int \log |f'| d\mu + u}$$

Also we get dimension results for target problems in the case of non-uniformly expanding transformations. In particular, for the Manneville-Pomeau transformation  $F(x) = x + x^{1+\alpha} \mod 1$  with  $0 < \alpha < 1$ . If we define

$$\tilde{u} = \begin{cases} u, & \text{if } x \neq 0 \\ (1 - \alpha)u, & \text{if } x = 0. \end{cases}$$

with  $u := -\lim_{n \rightarrow \infty} \frac{1}{n} \log r_n < \infty$ , then we get the following (see theorem 7.2):

**Theorem E.** *Let  $\alpha < T \leq 1$  be the unique solution of  $P_{top}(-t \log |F'|) = t\tilde{u}$ , then*

$$\text{Dim} \left\{ y \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{|F(y) - x|}{r_n} = 0 \right\} = T \geq h_\mu / (h_\mu + \tilde{u})$$

with  $h_\mu$  the entropy of the ACIP measure of  $F$ .

It is interesting to notice that we have bigger Hausdorff dimension for the case  $x = 0$  (the point with  $T'(x) = 1$ ).

### 1.3 A Hungerford lemma for Cantor-like sets

The kind of Cantor-like sets that we have managed to deal with in this paper have the peculiarity that the ratio between the  $\hat{\mu}$ -measure of a parent and his children increase wildly. This, in general, would imply zero dimension. However, if we add a nice property which guarantees that the sets of the same generation with the same parent (i.e brothers) are separated enough, then we get positive dimension. We think that the study of the dimension of Cantor-like sets with this characteristic and the relation with the thermodynamic formalisms is interesting by itself, and it would be useful in other contexts. For this reason, in section 4 we present a general description, as pattern subsets in the symbolic space, of sets with this structure and we prove some results (with the flavor of the classical Hungerford lemma, see e.g [39]) on lower estimates of their dimension. See corollaries 4.2 and 4.3 for the Gibbs case, the simplest one.

### 1.4 Outline of the paper.

In section 2 we introduce (following Carathéodory approach) the different dimensions that we use along this paper and we compare them. In section 3 we include several results on thermodynamic formalisms; we also define a mixing property for local weak  $\sigma$ -Gibbs measures which implies a fixed proportion of good points in 0-cylinders. In section 4 we introduce a family of Cantor-like sets in the symbolic space and state some Hungerford dimension results. The target problem for the shift transformation is the content of section 5. In sections 6 and 7 we consider the target problems (ball and block) for Markov transformations and intermittent systems respectively.

*A few words about notation.* For two sequences  $\{a_n\}$  and  $\{b_n\}$  we write  $a_n \asymp b_n$  if for some fixed positive absolute constant  $c$  we have that  $c^{-1}b_n \leq a_n \leq cb_n$  for  $n \geq 1$ . If  $\lambda$  and  $\mu$  are two measures on  $X$ , we will say that  $\lambda$  and  $\mu$  are comparable on a subset  $Y \subset X$  if for some positive constant  $C = C(Y)$  we have that  $C^{-1}\mu(Z) \leq \lambda(Z) \leq C\mu(Z)$  for any Borel subset  $Z$  of  $Y$ . By convention, if  $A$  denote the empty subset in the set  $[0, 1]$ , then  $\sup(A) = 0$  and  $\inf(A) = 1$ .

## 2 Carathéodory dimensions

Let  $(X, d)$  be a separable metric space and let denote by  $\text{diam}(\cdot)$  the diameter. Next we recall the Carathéodory's construction of Borel measures (see e.g. [34]).

Let  $\mathcal{F}$  be a family of subsets of  $X$ , and  $\varphi$  a non-negative function on  $\mathcal{F}$  verifying:

- (i) For every  $\varepsilon > 0$  there are  $F_1, F_2, \dots \in \mathcal{F}$  such that  $\text{diam}(F_i) \leq \varepsilon$  and  $X = \bigcup_{i=1}^{\infty} F_i$ .
- (ii) For every  $\varepsilon > 0$  there is  $F \in \mathcal{F}$  such that  $\text{diam}(F) \leq \varepsilon$  and  $\varphi(F) \leq \varepsilon$ .

We will say that  $(\mathcal{F}, \varphi)$  is a *Carathéodory's pair*.

For  $0 < \varepsilon \leq \infty$  and  $A \subset X$  we define the regular Borel measure

$$M_{\mathcal{F}, \varphi}(A) = \lim_{\varepsilon \rightarrow 0} M_{\mathcal{F}, \varphi, \varepsilon}(A) \tag{5}$$



with

$$M_{\mathcal{F}, \varphi, \varepsilon}(A) = \inf \left\{ \sum_i \varphi(F_i) : A \subset \cup_i F_i, \text{diam } F_i \leq \varepsilon, F_i \in \mathcal{F} \right\}$$

In particular, we can consider a Carathéodory's pair  $(\mathcal{F}, \varphi)$  with  $\varphi(\cdot) = \psi(\cdot)^\alpha$  with  $\psi$  a non-negative function on  $\mathcal{F}$  and  $0 \leq \alpha < \infty$ . It is not difficult to check that if the function  $\psi$  verifies that

$$\sup \{ \psi(F) : F \in \mathcal{F}, \text{diam}(F) \leq \varepsilon \} \rightarrow 0 \quad \text{when } \varepsilon \rightarrow 0 \quad (6)$$

then the regular Borel measures  $M_{\mathcal{F}, \psi(\cdot)^\alpha}$  satisfies for all  $A \subset X$

$$\begin{aligned} M_{\mathcal{F}, \psi^\alpha}(A) < \infty &\implies M_{\mathcal{F}, \psi^\beta}(A) = 0 && \text{for } \beta > \alpha, \\ M_{\mathcal{F}, \psi^\alpha}(A) > 0 &\implies M_{\mathcal{F}, \psi^\beta}(A) = \infty && \text{for } \beta < \alpha, \end{aligned}$$

Hence we can define the  $(\mathcal{F}, \varphi)$ -dimension as

$$\text{Dim}_{\mathcal{F}, \psi}(A) := \inf \{ \alpha : M_{\mathcal{F}, \psi^\alpha}(A) = 0 \} = \sup \{ \alpha : M_{\mathcal{F}, \psi^\alpha}(A) > 0 \}.$$

If  $\mathcal{F}$  is the collection of all open balls in  $X$  we will refer to  $\text{Dim}_{\mathcal{F}, \psi}(A)$  as the  $\varphi$ -spherical dimension. If  $\mathcal{F} = \{F : F \subset X\}$  and  $\varphi(\cdot) = (\text{diam}(\cdot))^\alpha$  for some  $0 \leq \alpha < \infty$ , the measure  $M_{\mathcal{F}, \varphi}$  is called the  $\alpha$ -dimensional Hausdorff measure and we will denote by  $H_{(\text{diam})^\alpha}$ ; moreover,  $\text{Dim}_{\mathcal{F}, \text{diam}}$  is the usual Hausdorff dimension, for simplicity we will denote it by  $\text{Dim}$ .

## 2.1 Topological Markov chains and $\hat{\mu}$ dimension.

Let  $(\Sigma_A^{\mathcal{I}}, d, \sigma)$  be a (one-sided) topological Markov chain, see introduction. Recall that, for  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathcal{I}}$ , we denote the  $n$ -cylinder  $C_{w_0 w_1 \dots w_n}$  by  $C(n, w)$ . Also we will use  $w|_n$  to indicate the finite sequence  $(w_0, w_1, \dots, w_n)$ , and  $\Sigma_A^{\mathcal{I}}|_n := \{w|_n : w \in \Sigma_A^{\mathcal{I}}\}$ .

A topological Markov chain  $\Sigma_A^{\mathcal{I}}$  is *topologically mixing* if for every  $a, b \in \mathcal{I}$  there exists  $N = N(a, b)$  such that: for all  $n \geq N$  there exists  $w|_n \in \Sigma_A^{\mathcal{I}}|_n$  with  $w_0 = a$  and  $w_n = b$ . Along this paper we will assume this property.

A topological Markov chain  $\Sigma_A^{\mathcal{I}}$  satisfies the *big images* (BI) property if there exist a finite set of symbols  $\mathcal{I}_0 \subset \mathcal{I}$  such that: for all symbol  $i \in \mathcal{I}$  there exists  $k \in \mathcal{I}_0$  such that  $C_{ik} \neq \emptyset$ . This condition is called the big images property because if  $m$  is some finite measure supported in  $\Sigma_A^{\mathcal{I}}$ , then is equivalent to the condition  $\inf \{m(\sigma(C_i)) : i \in \mathcal{I}\} > 0$ .

A topological Markov chain satisfies the *big images and preimages* (BIP) property if there exist a finite set of symbols  $\mathcal{I}_0 \subset \mathcal{I}$  such that: for all symbol  $i \in \mathcal{I}$  there exist  $k, \ell \in \mathcal{I}_0$  such that  $C_{kil} \neq \emptyset$ .

**Definition 2.1.** Given a finite atomless measure  $\hat{\mu}$  in the topological Markov chain  $(\Sigma_A^{\mathcal{I}}, d)$ , we define the  $\hat{\mu}$ -dimension of any set  $E \subset \Sigma_A^{\mathcal{I}}$  as the  $(\mathcal{F}, \hat{\mu})$ -(Carathéodory)-dimension with  $\mathcal{F}$  the set of all cylinders in  $\Sigma_A^{\mathcal{I}}$ . For simplicity we will write  $\text{Dim}_{\hat{\mu}}(E)$  instead of  $\text{Dim}_{\mathcal{F}, \hat{\mu}}(E)$ .

Notice that condition (6) holds due to the atomless. We will use  $\text{Dim}_{\hat{\mu}}(E)$  to study the size of sets of zero  $\hat{\mu}$ -measure.

## 2.2 Grid dimension vs Hausdorff dimension in $[0, 1]$

Besides of the usual Hausdorff dimension in the interval  $[0, 1]$ , which we denote by  $\text{Dim}$ , we will also consider the *grid dimension* defined following Carathéodory's construction.

**Definition 2.2.** A grid inside  $[0, 1]$  is a sequence  $\Pi = \{\mathcal{P}_n\}$  of collections  $\mathcal{P}_n$  of subsets of  $[0, 1]$  such that:

- (i) For all  $n$ , if  $P, P' \in \mathcal{P}_n$  with  $P \neq P'$ , then  $P \cap P' = \emptyset$ .
- (ii) For all  $P_n \in \mathcal{P}_n$  there exists a unique  $P_{n-1} \in \mathcal{P}_{n-1}$  such that  $P_n \subset P_{n-1}$ .
- (iii)  $\sup_{P \in \mathcal{P}_n} \text{diam}(P) \rightarrow 0$  as  $n \rightarrow \infty$ .

A grid define the sets

$$X_{\Pi} := \bigcap_n \bigcup_{P \in \mathcal{P}_n} \text{cl}(P) \quad \text{and} \quad X_{\Pi}^{1-1} := \bigcap_n \bigcup_{P \in \mathcal{P}_n} P.$$

Given a grid  $\Pi = \{\mathcal{P}_n\}$ , we consider the collection

$$\mathcal{F} := \{\text{cl}(P) \cap X_{\Pi} : P \in \mathcal{P}_n \text{ for some } n\}.$$

The grid dimension in  $X_{\Pi}$  is defined as the  $(\mathcal{F}, \text{diam})$ -dimension; we will use  $\text{Dim}_{\Pi}$  to denote it.

**Remark 2.1.** Notice that for all  $E \subset X_{\Pi}$ ,  $\text{Dim}(E) \leq \text{Dim}_{\Pi}(E) \leq 1$

**Proposition 2.1.** Let  $E \subset X_{\Pi}$  such that

$$\#\{P \in \mathcal{P}_0 : \text{cl}(P) \cap E \neq \emptyset\} < \infty. \quad (7)$$

and let us suppose that there exist  $\gamma \geq 0$  and  $C > 0$  such that for all  $P_n \in \mathcal{P}_n$  with  $\text{cl}(P_n) \cap E \neq \emptyset$

$$\frac{\text{diam}(P_{n-1})^{1+\gamma}}{\text{diam}(P_n)} \leq C, \quad \text{where } P_n \subset P_{n-1} \in \mathcal{P}_{n-1}. \quad (8)$$

Then, for  $\gamma/(1+\gamma) < \alpha \leq 1$

$$M_{\mathcal{F}, \text{diam}^{\alpha}}(E) \leq C_0 H_{\text{diam}^{\alpha-\gamma(1-\alpha)}}(E) \quad (9)$$

with  $C_0$  a positive constant. Here  $H_{\text{diam}^{\eta}}$  denotes the  $\eta$ -dimensional Hausdorff measure. In particular,

$$\text{Dim}_{\Pi}(E) \leq \text{Dim}(E) + \gamma(1 - \text{Dim}_{\Pi}(E))$$

**Remark 2.2.** Notice that if  $E$  satisfies (7), and for all  $\gamma > 0$  there exists  $C$  such that (8) holds, then  $\text{Dim}_{\Pi}(E) = \text{Dim}(E)$ .

*Proof.* Let  $I$  be an interval in  $[0, 1]$  with small radius and such that  $I \cap E \neq \emptyset$ . Then there exists a collection  $\{P_i = P(n_i, x_i)\}$  such that  $I \cap E \subset \cup_i \text{cl}(P_i) \cap X_{\Pi}$  and

$$\text{diam}(\text{cl}(P(n_i, x_i)) \cap X_{\Pi}) \leq \text{diam}(I) \quad \text{but} \quad \text{diam}(I) < \text{diam}(\text{cl}(P(n_i - 1, x_i)) \cap X_{\Pi}),$$

and

$$\sum_i \text{diam}(\text{cl}(P_i) \cap X_{\Pi}) \leq 3 \text{diam}(I)$$

Notice that from (7) and (8) we have that there exists  $K > 0$  such that for all for all  $P_n \in \mathcal{P}_n$  with  $\text{cl}(P_n) \cap E \neq \emptyset$

$$\text{diam}(\text{cl}(P_n) \cap X_{\Pi}) \geq \left(\frac{1}{C^{1/\gamma}}\right)^{(1+\gamma)^n - 1} \text{diam}(\text{cl}(P_0) \cap X_{\Pi})^{(1+\gamma)^n} \geq C^{1/\gamma} K^{(1+\gamma)^n}.$$

Hence, if  $I$  has small radius then by the above inequality we have that the generations  $n_i$  are large. Also from (8) we have that

$$\text{diam}(\text{cl}(P_i) \cap X_{\Pi}) = \text{diam}(\text{cl}(P(n_i, x_i)) \cap X_{\Pi}) \geq C' \text{diam}(\text{cl}(P(n_i - 1, x_i)) \cap X_{\Pi})^{1+\gamma} > C'' \text{diam}(I)^{1+\gamma}.$$

Hence, for  $0 \leq \alpha \leq 1$

$$\sum_i (\text{diam}(\text{cl}(P_i) \cap X_{\Pi}))^{\alpha} = \sum_i \frac{\text{diam}(\text{cl}(P_i) \cap X_{\Pi})}{(\text{diam}(\text{cl}(P_i) \cap X_{\Pi}))^{1-\alpha}} \leq C \frac{\text{diam}(I)}{\text{diam}(I)^{(1+\gamma)(1-\alpha)}} = C \text{diam}(I)^{\alpha-\gamma(1-\alpha)},$$

and we get the inequality (9).  $\square$

### 2.3 Shift modeled transformations in $[0, 1]$ , grid and $\widehat{\mu}$ dimensions

Let  $\lambda$  denote the Lebesgue measure in the interval  $[0, 1]$  and let  $f : [0, 1] \rightarrow [0, 1]$  be a map with the property that there exists a finite or numerable family  $\mathcal{P}_0 = \{P_i^0\}_{i \in \mathcal{I}}$  of disjoint open intervals in  $[0, 1]$  such that:

- (a)  $\lambda([0, 1] \setminus \cup_j P_j^0) = 0$ .
- (b) If  $P_i^0, P_j^0 \in \mathcal{P}_0$  and  $f(P_i^0) \cap P_j^0 \neq \emptyset$ , then  $P_j^0 \subset f(P_i^0)$ .
- (c) For each  $j$ , the map  $f : P_j^0 \rightarrow f(P_j^0)$  is continuous and injective.
- (d)  $\sup_{P \in \mathcal{P}_n} \text{diam}(P) \rightarrow 0$  as  $n \rightarrow \infty$  with

$$\mathcal{P}_n = \bigcup_{P_i^0 \in \mathcal{P}_0} \{(f|_{P_i^0})^{-1}(P_j) : P_j \in \mathcal{P}_{n-1}, P_j \subset f(P_i^0)\} = \bigvee_{j=0}^n f^{-j}(\mathcal{P}_0).$$

It is clear that  $\mathbf{\Pi} = \{\mathcal{P}_n\}$  is a grid in  $[0, 1]$ , and moreover from (a) and (b) it follows that  $\lambda(X_{\mathbf{\Pi}}) = \lambda(X_{\mathbf{\Pi}}^{1-1}) = 1$ . From (c) we have that the elements of  $\mathcal{P}_n$  are open intervals.

The transformation  $f$  can be modeled by the left shift acting on a topological Markov chain: Let  $\mathcal{I}$  be the finite or numerable set indexing the initial partition  $\mathcal{P}_0 = \{P_i^0\}_{i \in \mathcal{I}}$  and let  $A = (a_{i,j})$  be the  $\mathcal{I} \times \mathcal{I}$  matrix with entries 0 and 1 defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } f(P_i^0) \cap P_j^0 \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

The map  $\pi : \Sigma_A^{\mathcal{I}} \rightarrow X_{\mathbf{\Pi}}$  defined by  $\pi((i_0, i_1, \dots)) = x$ , where  $x$  is the unique point such that

$$\{x\} := \bigcap_{n=0}^{\infty} \text{cl}(f^{-n}(P_{i_n}^0)) = \bigcap_{n=0}^{\infty} \text{cl}(P_{i_0} \cap f^{-1}(P_{i_1}^0) \cap \dots \cap f^{-n}(P_{i_n}^0)),$$

is continuous and  $f \circ \pi = \pi \circ \sigma$ . If  $P_{i_0 i_1 \dots i_n}$  denote the interval in  $\mathcal{P}_n$  defined by

$$P_{i_0 i_1 \dots i_n} = \{x \in [0, 1] : x \in P_{i_0}^0, f(x) \in P_{i_1}^0, \dots, f^n(x) \in P_{i_n}^0\},$$

then  $\pi(C_{i_0 i_1 \dots i_n}) = \text{cl}(P_{i_0 i_1 \dots i_n}) \cap X_{\mathbf{\Pi}}$ . Notice also, that the map  $\pi$  is injective in  $\pi^{-1}(X_{\mathbf{\Pi}}^{1-1})$ . For any  $x \in X_{\mathbf{\Pi}}^{1-1}$  there is a uniquely determinated block sequence  $\{P_n\}$  with  $P_n \in \mathcal{P}_n$  and  $P_{n+1} \subset P_n$  such that  $\cap_n \text{cl}(P_n) = \{x\}$ , we will denote  $P_n$  by  $P(n, x)$ . If  $x \in X_{\mathbf{\Pi}} \setminus X_{\mathbf{\Pi}}^{1-1}$ , the sequence  $\{P_n\}$  is not uniquely determinated by  $x$ , from now on, for each  $x$  we choose  $\{P_n\}$  and we denote  $P_n$  by  $P(n, x)$ , i.e if there exists  $w, w'$  such that  $\pi(w) = \pi(w') = x$  we choose one of them, say  $w = (w_0, w_1, \dots)$ , and define  $P(n, x) = P_{w_0 w_1 \dots w_n}$ .

We will refer to  $(\Sigma_A^{\mathcal{I}}, d, \sigma)$  as the *symbolic representation* of  $f$ .

Let  $\widehat{\mu}$  be a finite atomless measure in the topological Markov chain  $(\Sigma_A^{\mathcal{I}}, d)$  such that  $\widehat{\mu} \circ \pi \asymp \lambda$  in  $\text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}$  for each interval  $P_i^0 \in \mathcal{P}_0$ .

Notice that  $X_{\mathbf{\Pi}} \setminus X_{\mathbf{\Pi}}^{1-1} \subset \bigcup_{n \geq N} \mathcal{B}_n$ , with  $\mathcal{B}_n$  the set of points in  $X_{\mathbf{\Pi}}$  belonging to the boundary of some interval of the family  $\mathcal{P}_n$ , and also  $\#(\pi^{-1}(x)) \leq 2$ . The following lemma is an easy consequence of these facts and we will not include the proof.

**Lemma 2.1.** *Let  $\mathcal{F}_1 = \{\text{cl}(P) \cap X_{\mathbf{\Pi}} : P \in \mathcal{P}_n \text{ for some } n\}$ ,  $\mathcal{F}_2 = \{C \subset \Sigma_A^{\mathcal{I}} : C \text{ a cylinder}\}$ , and  $\alpha > 0$ . Then*

$$M_{\mathcal{F}_1, \text{diam}^\alpha}(X_{\mathbf{\Pi}} \setminus X_{\mathbf{\Pi}}^{1-1}) = M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma_A^{\mathcal{I}} \setminus \pi^{-1}(X_{\mathbf{\Pi}}^{1-1})) = 0$$

We have the following relation between grid and  $\widehat{\mu}$ -dimensions.

**Lemma 2.2.**  $\text{Dim}_{\mathbf{\Pi}}(\pi(\Sigma)) = \text{Dim}_{\widehat{\mu}}(\Sigma)$  for all  $\Sigma \subset \Sigma_A^{\mathcal{I}}$ .

*Proof.* The blocks  $P_{i_0 i_1 \dots i_n}$  in  $\mathcal{P}_n$  are intervals contained in the initial intervals of  $\mathcal{P}_0$  and since  $\widehat{\mu} \circ \pi \asymp \lambda$  in  $\text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}$  for each interval  $P_i^0 \in \mathcal{P}_0$  we have that

$$\text{diam}(P_{i_0 i_1 \dots i_n}) = \lambda(P_{i_0 i_1 \dots i_n}) = \lambda(\text{cl}(P_{i_0 i_1 \dots i_n}) \cap X_{\mathbf{\Pi}}) \asymp \widehat{\mu} \circ \pi(\text{cl}(P_{i_0 i_1 \dots i_n}) \cap X_{\mathbf{\Pi}}) \quad (10)$$

with constants depending on the initial block  $P_{i_0}^0$ . We recall that  $\lambda(X_{\mathbf{\Pi}}) = 1$ .

Let  $\mathcal{F}_1 = \{\text{cl}(P) \cap X_{\mathbf{\Pi}} : P \in \mathcal{P}_n \text{ for some } n\}$  and  $\mathcal{F}_2 = \{C \subset \Sigma_A^{\mathbb{Z}} : C \text{ a cylinder}\}$ . First, notice that from lemma 2.1, since  $(\pi(\Sigma) \cap \pi(C_i)) \setminus \pi(\Sigma \cap C_i) \subset X_{\mathbf{\Pi}} \setminus X_{\mathbf{\Pi}}^{1-1}$ , we have that

$$M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma) \cap \pi(C_i)) = M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma \cap C_i)) \quad (11)$$

and

$$M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i) = M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i \cap \pi^{-1}(X_{\mathbf{\Pi}}^{1-1})) \quad (12)$$

Next, we will see that

$$M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i) = 0 \iff M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma) \cap \text{cl}(P_i^0)) = 0 \quad (13)$$

( $\implies$ ) If  $\Sigma \cap C_i \subset \bigcup_{F_2 \in \mathcal{D}_2} F_2 \subset C_i$  with  $\mathcal{D}_2 \subset \mathcal{F}_2$ , then

$$\pi(\Sigma \cap C_i) \subset \bigcup_{F_1 \in \mathcal{D}_1} F_1 \subset \text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}$$

with  $\mathcal{D}_1 = \{\pi(F_2) : F_2 \in \mathcal{D}_2\} \subset \mathcal{F}_1$ . From (10) we have that  $\text{diam}(F_1) \asymp \widehat{\mu}(F_2)$  for  $F_1 = \pi(F_2)$ , and therefore if  $M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i) = 0$  then  $M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma \cap C_i)) = 0$ . By (11) we get  $M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma) \cap \text{cl}(P_i^0)) = 0$ .

( $\impliedby$ ) If  $\pi(\Sigma) \cap \text{cl}(P_i^0) \subset \bigcup_{F_1 \in \mathcal{D}_1} F_1 \subset \text{cl}(P_i^0)$  with  $\mathcal{D}_1 \subset \mathcal{F}_1$ , then

$$\pi^{-1}[\pi(\Sigma) \cap \text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}^{1-1}] \subset \bigcup_{F_1 \in \mathcal{D}_1} \pi^{-1}(F_1 \cap X_{\mathbf{\Pi}}^{1-1})$$

But  $\pi^{-1}[\text{cl}(P_{i_0 i_1 \dots i_n}) \cap X_{\mathbf{\Pi}}^{1-1}] \subset C_{i_0 i_1 \dots i_n}$  and so from (10) we get that

$$\pi^{-1}[\pi(\Sigma) \cap \text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}^{1-1}] \subset \bigcup_{F_2 \in \mathcal{D}_2} F_2 \quad \text{with } \mathcal{D}_2 \subset \mathcal{F}_2$$

and

$$\sum_{F_1 \in \mathcal{D}_1} \text{diam}^\alpha(F_1) = \sum_{F_2 \in \mathcal{D}_2} \widehat{\mu}^\alpha(F_2).$$

Notice also that

$$\Sigma \cap C_i \cap \pi^{-1}(X_{\mathbf{\Pi}}^{1-1}) \subset \pi^{-1}[\pi(\Sigma) \cap \text{cl}(P_i^0) \cap X_{\mathbf{\Pi}}^{1-1}]$$

Therefore if  $M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma) \cap \text{cl}(P_i^0)) = 0$  then  $M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i \cap \pi^{-1}(X_{\mathbf{\Pi}}^{1-1})) = 0$ , and from (12)  $M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i) = 0$ .

Finally, notice that the statement follows from (13) since

$$M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma)) = 0 \iff M_{\mathcal{F}_1, \text{diam}^\alpha}(\pi(\Sigma) \cap \text{cl}(P_i^0)) = 0 \quad \text{for all } P_i^0 \in \mathcal{P}^0$$

$$M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma) = 0 \iff M_{\mathcal{F}_2, \widehat{\mu}^\alpha}(\Sigma \cap C_i) = 0 \quad \text{for all cylinder } C_i$$

□

### 3 Thermodynamic formalism for countable Markov shifts

Let  $\Sigma_A^{\mathcal{I}}$  be a topologically mixing Markov chain. The basic notion of the thermodynamic formalism is the *topological pressure* which was introduced by Ruelle [40] and Walters [54] for continuous transformations acting on a compact space. In the case of finite alphabet, for any continuous function  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$

$$P_{top}(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{w|_n \in \Sigma_A^{\mathcal{I}}|_n} \exp \left[ \sup_{z \in C(n,w)} \sum_{j=0}^{n-1} \phi(\sigma^j(z)) \right] \quad (14)$$

For topologically mixing countable Markov chains, O. Sarig introduced in [43] the Gurevich pressure for some appropriate potentials. First let us recall some regularity conditions for the potentials.

A potential  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  is said to have *summable variations* if  $\sum_{n \geq 2} V_n(\phi) < \infty$  where the variation  $V_n(\phi)$  is defined by

$$V_n(\phi) := \sup\{|\phi(w) - \phi(w')| : w = (i_0, i_1, \dots), w' = (j_0, j_1, \dots) \in \Sigma_A^{\mathcal{I}}, i_k = j_k, 0 \leq k \leq n-1\}.$$

The potential  $\phi$  satisfies the *Walter condition* if for every  $k \geq 1$ ,

$$\sup_{n \geq 1} [V_{n+k}(\sum_{j=0}^{n-1} \phi \circ \sigma^j)] < \infty \quad \text{and} \quad \sup_{n \geq 1} [V_{n+k}(\sum_{j=0}^{n-1} \phi \circ \sigma^j)] \rightarrow 0 \text{ as } k \rightarrow \infty;$$

the second condition implies the first one when the alphabet is finite. Notice that Walter condition is weaker than summable variations.

Moreover, we will say that  $\phi$  is *weakly Hölder continuous* iff there exists  $A > 0$  and  $\theta \in (0, 1)$  such that for all  $n \geq 2$ ,  $V_n(\phi) \leq A\theta^n$ . Weak Hölder continuity is stronger than summable variation.

For a potential  $\phi$  with summable variations O. Sarig defined the Gurevich pressure,  $P_G(\phi)$ , as

$$P_G(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{\substack{w=(i_0, i_1, \dots) \in \Sigma_A^{\mathcal{I}} \\ i_0=i, \sigma^n(w)=w}} \exp \left[ \sum_{j=0}^{n-1} \phi(\sigma^j(w)) \right] \quad \text{with } i \in \mathcal{I}$$

This pressure is a generalization of the Gurevich entropy  $h_G(\sigma)$  [18], since  $P_G(0) = h_G(\sigma)$ . This limit does not depend on  $i$ , it is never  $-\infty$  and has the following convexity property for potentials  $\phi, \psi$  of summable variations:  $P_G(t\phi + (1-t)\psi) \leq tP_G(\phi) + (1-t)P_G(\psi)$  for all  $0 \leq t \leq 1$ .

#### 3.1 Weak Gibbs measures

We will use the following definition of a local weak Gibbs measure.

**Definition 3.1.** A probability  $\hat{\mu}$  on  $\Sigma_A^{\mathcal{I}}$  is called a *local weak Gibbs measure* for the potential  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  if there exists a constant  $P$  and a sequence  $\{K_n\}$  of positive numbers with  $1 \leq K_n \leq K_{n+1}$  such that  $\lim_{n \rightarrow \infty} \frac{1}{n} \log K_n = 0$  and for  $\hat{\mu}$ -a.e.  $z \in C_{i_0 i_1 \dots i_{n-1}}$

$$\frac{1}{c(i_0)K_n} \leq \frac{\hat{\mu}(C_{i_0 i_1 \dots i_{n-1}})}{\exp(-nP + \sum_{j=0}^{n-1} \phi \circ \sigma^j(z))} \leq c(i_0)K_n \quad (15)$$

with  $c(i_0) \geq 1$  a constant depending on  $i_0$ . If the measure  $\hat{\mu}$  is  $\sigma$ -invariant, we will say that  $\hat{\mu}$  is a *local weak  $\sigma$ -Gibbs measure*.

If  $\sup_{i \in \mathcal{I}} c(i) < \infty$ , then  $\hat{\mu}$  is called a *weak Gibbs measure*. If  $\sup K_n < \infty$  and (15) holds for all  $z \in C_{i_0 i_1 \dots i_{n-1}}$ , then  $\hat{\mu}$  is called a *local Gibbs measure*. A *local Gibbs measure* is a Gibbs measure if  $\sup_{i \in \mathcal{I}} c(i) < \infty$ .

**Remark 3.1.** If the alphabet  $\mathcal{I}$  is infinite numerable but  $\phi$  has summable variation then the constant  $P$  is the Gurevich pressure  $P_G(\phi)$ . Moreover, if  $\sum_{n \geq 1} V_n(\phi) < \infty$ , then (15) holds for all  $z \in C_{i_0 i_1 \dots i_{n-1}}$  with an appropriate new constant  $c(i_0)$ .

We will refer later to the following property of a local weak  $\sigma$ -Gibbs measure

**Remark 3.2.** If  $\hat{\mu}$  is a local weak  $\sigma$ -Gibbs measure, then for  $m > n$

$$\frac{1}{s_{n,m}} \frac{\hat{\mu}(\sigma^n(C(m, z)))}{\hat{\mu}(\sigma^n(C(n, z)))} \leq \frac{\hat{\mu}(C(m, z))}{\hat{\mu}(C(n, z))} \leq s_{n,m} \frac{\hat{\mu}(\sigma^n(C(m, z)))}{\hat{\mu}(\sigma^n(C(n, z)))}$$

with  $s_{n,m}(z) := c(z_0)^2 c(z_n)^2 K_1 K_{m-n+1} K_{n+1} K_{m+1} \leq c(z_0)^2 c(z_n)^2 K_{m+1}^4$ .

For  $m = n - 1$  we have

$$\frac{1}{s_{n,n-1}} \frac{1}{\hat{\mu}(\sigma^n(C(n, z)))} \leq \frac{\hat{\mu}(C(n-1, z))}{\hat{\mu}(C(n, z))} \leq s_{n,n-1} \frac{1}{\hat{\mu}(\sigma^n(C(n, z)))}$$

with  $s_{n,n-1}(z) := c(z_0)^2 c(z_n) K_1 K_n K_{n+1} \leq c(z_0)^2 c(z_n) K_{n+1}^3$

*Proof.* From Definition 3.1 we have that for  $\hat{\mu}$ -a.e.  $\tilde{z} \in C(m, z)$

$$\frac{1}{c(z_0)^2 K_{m+1} K_{n+1}} \leq \frac{\hat{\mu}(C(m, z))}{\hat{\mu}(C(n, z))} \frac{1}{e^{-(m-n)P} \exp(\sum_{j=n+1}^m \phi \circ \sigma^j(\tilde{z}))} \leq c(z_0)^2 K_{m+1} K_{n+1} \quad (16)$$

and for  $\hat{\mu}$ -a.e.  $w \in \sigma^n(C(m, z))$

$$\frac{1}{c(z_n)^2 K_{m-n+1} K_1} \leq \frac{\hat{\mu}(\sigma^n(C(m, z)))}{\hat{\mu}(\sigma^n(C(n, z)))} \frac{1}{e^{-(m-n)P} \exp(\sum_{j=1}^{m-n} \phi \circ \sigma^j(w))} \leq c(z_n)^2 K_{m-n+1} K_1 \quad (17)$$

If  $W$  denotes the set of points in  $\sigma^n(C(m, z))$  verifying (17), then  $\sigma^{-n}(W)$  is a subset of  $C(m, z)$  with measure  $\hat{\mu}(\sigma^{-n}(W)) = \hat{\mu}(W) > 0$ . Therefore, there exist  $\tilde{z}, w$  with  $\sigma^n(\tilde{z}) = w$  verifying the inequalities (16) and (17). The first result follows directly from these inequalities. The proof for the case  $m = n - 1$  is similar.  $\square$

The following equality is called the *variational principle of topological pressure*

$$P_{top}(\phi) = \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(\sigma) + \int \phi d\mu \right\}$$

where  $\mathcal{M}$  is the set of  $\sigma$ -invariant probability measures in  $\Sigma_A^{\mathcal{I}}$  and  $h_\mu(\sigma)$  denotes the entropy of  $\sigma$  with respect to  $\mu$ . A measure  $\mu \in \mathcal{M}$  is called an *equilibrium measure for  $\phi$*  iff

$$h_\mu(\sigma) + \int \phi d\mu = \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(\sigma) + \int \phi d\mu \right\}.$$

For finite alphabet, classical results of Bowen and Ruelle (see [8], [40]) shows that for any  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  with the Walters property, there is a unique equilibrium measure and this measure is a  $\sigma$ -Gibbs measure. The existence of equilibrium and weak Gibbs measures has been also studied for non Hölder potentials. See [17], [22], [58], [59], [23].

For infinite countable alphabet Sarig obtained a variational principle for the Gurevich pressure (for  $\phi$  with summable variations and  $\sup \phi < \infty$ ) and studied the existence of equilibrium and Gibbs measures. See also Mauldin and Urbanski approach [31]. We remark that in the non-compact case one can have equilibrium measures which are not Gibbs and also Gibbs measures which are not equilibrium measures.

Sarig proved a generalized version of Ruelle-Perron-Frobenius theorem (see [43], [46], [47]), involving the modes of recurrence of the potential. In particular, for  $(\Sigma_A^{\mathcal{I}}, \sigma)$  a topologically mixing Markov chain and  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  a potential with summable variations and finite Gurevich pressure, he proved that  $\phi$  is positive recurrent iff there are  $\lambda > 0$ , a positive continuous function  $h$ , and a



conservative measure  $\nu$  which is finite on cylinders, such that  $L_\phi h = \lambda h$ ,  $L^* \nu = \lambda \nu$ , and  $\int h d\nu = 1$ . In this case  $\lambda = \exp P_G(\phi)$ , and for every cylinder  $C$ ,

$$\lambda^{-n} L_\phi^n(\mathbb{1}_C)(x) \rightarrow h(x) \frac{\nu(C)}{\int h d\nu} \quad \text{as } n \rightarrow \infty.$$

uniformly in  $x$  on compact sets. Here  $L_\phi(f) := \sum_{\sigma(y)=x} e^{\phi(y)} f(y)$  is the Ruelle operator. We refers to the measure  $dm = h d\nu$  as the Ruelle-Perron-Frobenius (RPF) measure. Moreover, he proved that if  $\mathbf{m}$  is the RPF measure of a potential  $\phi$ , which is positive recurrent with summable variations and such that  $P_G(\phi) < \infty$  and  $\sup \phi < \infty$ , and the entropy of  $\mathbf{m}$  is finite, then  $\mathbf{m}$  is an equilibrium measure which is exact (whence ergodic and strong mixing).

We recall that a potential  $\phi : \Sigma_A^\mathbb{Z} \rightarrow \mathbb{R}$  with summable variations and finite Gurevich pressure  $P := P_G(\phi)$  is *positive recurrent* iff for some (hence all)  $i \in \mathcal{I}$

$$\sum_n e^{-nP} Z_n(\phi, i) = \infty \quad \text{and} \quad \sum_n n e^{-nP} Z_n^*(\phi, i) < \infty \quad \text{with}$$

$$Z_n(\phi, i) = \sum_{\substack{w=(i_0, i_1, \dots) \in \Sigma_A^\mathbb{Z} \\ i_0=i, \sigma^n(w)=w}} \exp \left[ \sum_{j=0}^{n-1} \phi \circ \sigma^j(w) \right], \quad Z_n^*(\phi, i) = \sum_{\substack{w=(i_0, i_1, \dots) \in \Sigma_A^\mathbb{Z} \\ i_0=i, i_1, \dots, i_{n-1} \neq i, \sigma^n(w)=w}} \exp \left[ \sum_{j=0}^{n-1} \phi \circ \sigma^j(w) \right]$$

Notice that

$$P_G(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\phi, i)$$

In [44] Sarig proved that if  $(\Sigma_A^\mathbb{Z}, \sigma)$  is a topologically mixing Markov chain and  $\phi : \Sigma_A^\mathbb{Z} \rightarrow \mathbb{R}$  satisfies the Walters condition then  $\phi$  has a  $\sigma$ -Gibbs measure (with  $P = P_G(\phi)$ ) iff  $\Sigma_A^\mathbb{Z}$  has the BIP property and  $\phi$  has finite Gurevich pressure and  $V_1(\phi) < \infty$ . He proved that if  $\Sigma_A^\mathbb{Z}$  has the BIP property, then any potential  $\phi$  with the Walters property and such that  $V_1(\phi) < \infty$  and  $P_G(\phi) < \infty$ , is positive recurrent. A related result on the existence of Gibbs measures was obtained in [30] by Mauldin and Urbanski.

### 3.2 Some mixing properties

Along this section we will assume that  $\hat{\mu}$  is a local weak  $\sigma$ -Gibbs measure with potential  $\phi$  such that  $\phi \in L^1(\hat{\mu})$ . In the case of infinite alphabet we will assume that  $\sum_{n \geq 1} V_n(\phi) < \infty$  so that remark 3.1 holds. In next two section to avoid misunderstanding we will use  $\mathbf{P}$  (the bold style) to denote the constant (pressure) in the definition of local weak  $\sigma$ -Gibbs measure.

**Definition 3.2.** Given  $\varepsilon > 0$  and  $M \in \mathbb{N}$  we denote by  $\text{Good}(M, \varepsilon)$  the set of points  $z \in \Sigma_A^\mathbb{Z}$  such that

$$e^{-(j+1)[\mathbf{P} - \int \phi d\hat{\mu} + \varepsilon]} < \hat{\mu}(C(j, z)) < e^{-(j+1)[\mathbf{P} - \int \phi d\hat{\mu} - \varepsilon]}, \quad \text{for all } j \geq M. \quad (18)$$

If  $\hat{\mu}$  is ergodic, as a consequence of Birkhoff's ergodic theorem and remark 3.1, we have that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \hat{\mu}(C(n, z)) = -\mathbf{P} + \int \phi d\hat{\mu} \quad \text{for } z \in \hat{\mu} - a.e. \quad (19)$$

Hence by Egoroff's theorem we get that

**Lemma 3.1.** If  $\hat{\mu}$  is ergodic, then given  $\varepsilon > 0$  and any 0-cylinder  $P_1$

$$\hat{\mu}(P_1 \cap \text{Good}(M, \varepsilon)) \rightarrow \hat{\mu}(P_1) \quad \text{as } M \rightarrow \infty$$

**Proposition 3.1.** Let  $P_1, P_2$  be two 0-cylinders, and given  $\varepsilon > 0$  let  $\mathcal{S}_N(M, \varepsilon)$  denote the collection of cylinders  $C(N, z)$  with  $z \in \text{Good}(M, \varepsilon)$  verifying

$$C(N, z) \subset P_1, \quad \sigma^N(C(N, z)) \subset P_2.$$

If  $\hat{\mu}$  is mixing then, for all  $M$  large enough (depending on  $P_1$  and  $\varepsilon$ ) and  $N$  large enough (depending on  $P_1$  and  $P_2$ ),

$$\hat{\mu}(\mathcal{S}_N(M, \varepsilon)) := \mu\left(\bigcup_{S \in \mathcal{S}_N(M, \varepsilon)} S\right) \geq \frac{1}{2} \hat{\mu}(P_1) \hat{\mu}(P_2). \quad (20)$$

*Proof.* Let  $\mathcal{P}$  denote the set of all 0-cylinders. We have that

$$\begin{aligned} \hat{\mu}(P_1) &= \hat{\mu}(\mathcal{S}_N(M, \varepsilon)) + \sum_{P \in \mathcal{P} \setminus \{P_2\}} \sum_{\substack{C(N, z) \subset P_1 \\ \sigma^N(C(N, z)) = P}} \hat{\mu}(C(N, z)) + \hat{\mu}(P_1 \setminus \text{Good}(M, \varepsilon)) \\ &\leq \hat{\mu}(\mathcal{S}_N(M, \varepsilon)) + \sum_{P \in \mathcal{P} \setminus \{P_2\}} \hat{\mu}(P_1 \cap \sigma^{-N}(P)) + \hat{\mu}(P_1 \setminus \text{Good}(M, \varepsilon)) \\ &= \hat{\mu}(\mathcal{S}_N(M, \varepsilon)) + \hat{\mu}(P_1) - \hat{\mu}(P_1 \cap \sigma^{-N}(P_2)) + \hat{\mu}(P_1 \setminus \text{Good}(M, \varepsilon)). \end{aligned}$$

But  $\lim_{M \rightarrow \infty} \hat{\mu}(\text{Good}(M, \varepsilon)) = 1$  by lemma 3.1 and because  $\hat{\mu}$  is mixing

$$\lim_{N \rightarrow \infty} \hat{\mu}(P_1 \cap \sigma^{-N}(P_2)) = \hat{\mu}(P_1) \hat{\mu}(P_2).$$

Hence, we get (20). □

**Remark 3.3.** If  $\hat{\mu}$  is mixing and there exist  $m_0 \in \mathbb{N}$  and  $0 < \eta < 1$  such that

$$\frac{\hat{\mu}(P_1 \cap \text{Good}(m_0, \varepsilon))}{\hat{\mu}(P_1)} \geq 1 - \eta,$$

then for all 0-cylinder  $P_2$  with  $\hat{\mu}(P_2) > 2\eta$

$$\hat{\mu}(\mathcal{S}_N(m_0, \varepsilon)) \geq \frac{1}{2} \hat{\mu}(P_1) \hat{\mu}(P_2).$$

for all  $N$  large enough (depending on  $P_1$  and  $P_2$ ).

*Proof.* Take in the above proof  $N$  large so that

$$\frac{\hat{\mu}(P_1 \cap \sigma^{-N}(P_2))}{\hat{\mu}(P_1) \hat{\mu}(P_2)} \geq \frac{1}{2} + \frac{\eta}{\hat{\mu}(P_2)}.$$

□

**Definition 3.3.** A collection  $\mathcal{P}$  of 0-cylinders is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good iff there exist  $m_0$  and  $0 < \eta < 1$  such that

$$\frac{\hat{\mu}(P \cap \text{Good}(m_0, \varepsilon))}{\hat{\mu}(P)} \geq 1 - \eta, \quad \text{for all } P \in \mathcal{P}.$$

**Remark 3.4.** Of course if  $\hat{\mu}$  is ergodic any finite collection is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good, since  $\hat{\mu}(P \cap \text{Good}(M, \varepsilon)) \rightarrow \hat{\mu}(P)$  when  $M \rightarrow \infty$ .

From remark 3.3 follows that:

**Corollary 3.1.** If  $\hat{\mu}$  is mixing and  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\mu$ -good, then for all  $P_1 \in \mathcal{P}$  and all 0-cylinder  $P_2$  with  $\mu(P_2) > 2\eta$

$$\hat{\mu}(\mathcal{S}_N(m_0, \varepsilon)) \geq \frac{1}{2} \hat{\mu}(P_1) \hat{\mu}(P_2)$$

for all  $N$  large enough (depending on  $P_1$  and  $P_2$ ).

### 3.3 Summable uniform rate of mixing

In this section we look for setting up a better mixing property which allow us to get that an infinite countable collection of 0-cylinders is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good.

**Definition 3.4.** We will say that  $\hat{\mu}$  has a summable uniform rate of mixing  $\Psi$  (of order 3) in a collection  $\mathcal{P}$  of 0-cylinders iff there exist a decreasing function  $\Psi : \mathbb{N} \rightarrow \mathbb{R}$  and a natural number  $\ell_0$  such that

(1) For all  $P_1 \in \mathcal{P}$  and for all pair of 0-cylinders  $P_2, P_3$ , we have for  $k \geq 0$

$$|\hat{\mu}(P_1 \cap \sigma^{-k}(P_2) \cap \sigma^{-(k+\ell)}(P_3)) - \hat{\mu}(P_1 \cap \sigma^{-k}(P_2)) \hat{\mu}(P_3)| \leq \Psi(\ell) \hat{\mu}(P_1 \cap \sigma^{-k}(P_2)) \hat{\mu}(P_3) \text{ for } \ell \geq \ell_0,$$

and for  $0 < \ell < \ell_0$  there exists a positive constant  $C$  such that

$$\hat{\mu}(P_1 \cap \sigma^{-k}(P_2) \cap \sigma^{-(k+\ell)}(P_3)) \leq C \hat{\mu}(P_1) \hat{\mu}(P_2 \cap \sigma^{-\ell}(P_3))$$

(2)  $\sum_{\ell} \Psi(\ell) < \infty$ .

**Remark 3.5.** Condition (1) implies (taking  $k = 0$  and  $P_1 = P_2$ ) that

$$|\hat{\mu}(P_1 \cap \sigma^{-\ell}(P_3)) - \hat{\mu}(P_1) \hat{\mu}(P_3)| \leq \Psi(\ell) \hat{\mu}(P_1) \hat{\mu}(P_3) \text{ for } \ell \geq \ell_0.$$

Also, by addition over all the of 0-cylinders  $P_3$  we have that

$$\hat{\mu}(P_1 \cap \sigma^{-k}(P_2)) \leq C \hat{\mu}(P_1) \hat{\mu}(P_2).$$

**Remark 3.6.** We recall (see [29]) that  $\mathcal{P}$  is called continued fraction mixing (with respect to  $\hat{\mu}$ ) if for all  $k$ -cylinder  $Q$  with  $\sigma^k(Q) \in \mathcal{P}$  and for all Borel set  $A$  we have that

$$\hat{\mu}(Q \cap \sigma^{-(k+1)}(A)) \leq c \hat{\mu}(Q) \hat{\mu}(A) \text{ for some constant } c,$$

and for  $\ell$  large enough, say  $\ell \geq \ell_1$ ,

$$|\hat{\mu}(Q \cap \sigma^{-(k+\ell)}(A)) - \hat{\mu}(Q) \hat{\mu}(A)| \leq \Psi(\ell) \hat{\mu}(Q) \hat{\mu}(A), \text{ with } \Psi(\ell) \rightarrow 0 \text{ as } \ell \rightarrow \infty.$$

Continued fraction mixing implies condition (1). If  $\Psi(\ell) = \theta^\ell$  for some  $0 < \theta < 1$ , this property is usually called exponentially continued fraction mixing, and we also have condition (2).

**Theorem 3.1.** Let  $\hat{\mu}$  be a local weak  $\sigma$ -Gibbs measure with potential  $\phi$  such that  $\phi \in L^2(\hat{\mu})$ , and  $V_1(\phi) < \infty$ . If  $\hat{\mu}$  has summable uniform rate of mixing of order 3 in a collection  $\mathcal{P}$  of 0-cylinders and the constants of localness of the measure  $\hat{\mu}$  in  $\mathcal{P}$  are upper bounded then, given  $\varepsilon > 0$ , for all  $M$  large enough,

$$\frac{\hat{\mu}(P \cap \text{Good}(M, \varepsilon))}{\hat{\mu}(P)} \geq 1 - \frac{c}{\varepsilon^2} \sum_{n \geq M} \frac{1}{n^2}, \text{ for all 0-cylinder } P \text{ in } \mathcal{P} \quad (21)$$

with  $c$  a positive constant.

**Corollary 3.2.**  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good.

*Proof.* Notice that

$$C_j \cap (\text{Good}(M, \varepsilon))^c \subset \bigcup_{n \geq M} \left\{ z \in C_j : \left| -\log \hat{\mu}(C(n, z)) - (n+1) \left[ \mathbf{P} - \int \phi d\hat{\mu} \right] \right| \geq (n+1)\varepsilon \right\}$$

And, by the definition of a local weak  $\sigma$ -Gibbs measure, we have for  $M$  large (depending on  $\varepsilon$  and  $\sup\{c(j) : j \in \mathcal{J}\}$ ) that

$$C_j \cap (\text{Good}(M, \varepsilon))^c \subset \bigcup_{n \geq M} \left\{ z \in C_j : \left| \sum_{k=0}^n \phi \circ \sigma^k(z) - (n+1) \int \phi d\hat{\mu} \right| \geq (n+1)\varepsilon/2 \right\}$$

Let  $Z_k(z) = \phi \circ \sigma^k(z)$ . In order to get (21) we prove that for  $\varepsilon$  small and all 0-cylinder  $P \in \mathcal{P}$

$$\hat{\mu} \left( \bigcup_{n \geq M} \{z \in P : \left| \sum_{k=0}^n Z_k(z) - \sum_{k=0}^n E[Z_k] \right| > (n+1)\varepsilon\} \right) \leq c \frac{\hat{\mu}(P)}{\varepsilon^2} \sum_{n \geq M} \frac{1}{n^2} \quad (22)$$

The argument of the proof follows the same line that the proof of the strong law of large numbers. We will prove that for  $m \leq n$

$$E \left[ \left( \sum_{k=m}^n Z_k - \sum_{k=m}^n E[Z_k] \right)^2 \chi_P \right] \leq c \hat{\mu}(P)(n-m+1) \quad (23)$$

where  $\chi_P$  denote the characteristic function on  $P$ . Then, by Chebyshev's inequality we have that for  $t > 0$

$$\mu\{z \in P : \left| \sum_{k=m}^n Z_k(z) - \sum_{k=m}^n E[Z_k] \right| > t\varepsilon\} < \frac{E[(\sum_{k=m}^n Z_k - \sum_{k=m}^n E[Z_k])^2 \chi_P]}{t^2 \varepsilon^2} \leq c \hat{\mu}(P) \frac{(n-m+1)}{t^2 \varepsilon^2}$$

The above inequality implies (22). Just notice that the set whose measure we are considering in (22) is a subset of the union on  $n \geq M$  of the sets

$$A_{n^2} = \{z \in P : \left| \sum_{k=0}^{n^2} Z_k(z) - \sum_{k=0}^{n^2} E[Z_k] \right| > (n^2+1)\frac{\varepsilon}{2}\}$$

and

$$\bigcup_{n^2 < \ell < (n+1)^2} \{z \in P : \left| \sum_{k=n^2+1}^{\ell} Z_k(z) - \sum_{k=n^2+1}^{\ell} E[Z_k] \right| > (n^2+1)\frac{\varepsilon}{2}\}.$$

By computation we have that

$$\begin{aligned} E \left[ \left( \sum_{k=m}^n Z_k - \sum_{k=m}^n E[Z_k] \right)^2 \chi_P \right] &= \sum_{k=m}^n (E[Z_k^2 \chi_P] - 2E[Z_k]E[Z_k \chi_P] + (E[Z_k])^2 \hat{\mu}(P)) \\ &+ 2 \sum_{m \leq k < j \leq n} (E[Z_k Z_j \chi_P] - E[Z_k]E[Z_j \chi_P] - E[Z_j]E[Z_k \chi_P] + E[Z_k]E[Z_j] \hat{\mu}(P)) \end{aligned}$$

Since  $\phi \in L^2(\hat{\mu})$  and  $V_1(\phi) < \infty$ , by using Remark 3.5 we get that  $|E[Z_k \chi_P]| \leq c \hat{\mu}(P)$  and  $E[Z_k^2 \chi_P] \leq c \hat{\mu}(P)$ . Hence,

$$\sum_{k=m}^n (E[Z_k^2 \chi_P] - 2E[Z_k]E[Z_k \chi_P] + (E[Z_k])^2 \hat{\mu}(P)) \leq c \hat{\mu}(P)(n-m+1) \quad (24)$$

By using again that:  $\phi \in L^1(\hat{\mu})$ ,  $V_1(\phi) < \infty$ , Remark 3.5 and the  $\sigma$ -invariance of  $\hat{\mu}$ , we obtain for  $k < j$  and  $j \geq \ell_0$  that

$$E[Z_k]E[Z_j] \hat{\mu}(P) - E[Z_k]E[Z_j \chi_P] = E[Z_k] \sum_C \left( \hat{\mu}(P) \int_{\sigma^{-j}(C)} Z_j d\hat{\mu} - \int_{P \cap \sigma^{-j}(C)} Z_j d\hat{\mu} \right) \leq c \Psi(j) \hat{\mu}(P)$$

where the sum is over all 0-cylinders  $C$ . Notice that from  $k < j < \ell_0$  we have

$$E[Z_k]E[Z_j] \hat{\mu}(P) - E[Z_k]E[Z_j \chi_P] \leq |E[Z_k]|(|E[Z_j]| + |E[Z_j \chi_P]|) \leq c \hat{\mu}(P)$$

Therefore by property (2) in Definition 3.4

$$\sum_{m \leq k < j \leq n} (E[Z_k]E[Z_j] \mu(P) - E[Z_k]E[Z_j \chi_P]) \leq c \hat{\mu}(P) \sum_{k=m}^n \sum_{j=k+1}^n \Psi(j) \leq c \hat{\mu}(P)(n-m+1) \quad (25)$$

In a similar way, but by using now property (1) in Definition 3.4, we get that for  $j - k \geq \ell_0$

$$\begin{aligned} E[Z_k Z_j \chi_P] - E[Z_k \chi_P] E[Z_j] &= \\ &= \sum_{C, C'} \int_{P \cap \sigma^{-k}(C) \cap \sigma^{-j}(C')} Z_j Z_k d\hat{\mu} - \left( \sum_C \int_{P \cap \sigma^{-k}(C)} Z_k d\hat{\mu} \right) \left( \sum_{C'} \int_{\sigma^{-j}(C')} Z_j d\hat{\mu} \right) \leq c \Psi(j - k) \hat{\mu}(P) \end{aligned}$$

where the sum is over all 0-cylinders  $C, C'$ . For  $k < j < k + \ell_0$ , by using property (1) of Definition 3.4, we have that

$$E[Z_k Z_j \chi_P] - E[Z_k \chi_P] E[Z_j] \leq c \hat{\mu}(P) E[|Z_0 Z_{j-k}|] + |E[Z_k \chi_P] E[Z_j]| \leq c' \hat{\mu}(P)$$

Therefore by property (2) of Definition 3.4

$$\sum_{m \leq k < j \leq n} (E[Z_k Z_j \chi_P] - E[Z_k \chi_P] E[Z_j]) \leq c \hat{\mu}(P) \sum_{k=m}^n \sum_{j=k+1}^n \Psi(j - k) \leq c \hat{\mu}(P) (n - m + 1) \quad (26)$$

The inequality (23) follows from (24), (25) and (26).  $\square$

## 4 Pattern subsets in the symbolic space

We want to define a subset of  $\Sigma_A^{\mathbb{Z}}$  with certain regular structure.

**Definition 4.1.** *Let  $A$  be a set of  $m$ -cylinders and  $B$  be a set of  $n$ -cylinders with  $m > n$ . We will say that  $A$  is finer than  $B$  iff each  $m$ -cylinder in  $A$  is contained in a  $n$ -cylinder in  $B$ . We will write  $A < B$ .*

**Definition 4.2.** *Let us consider two sequences  $\{\tilde{\mathcal{J}}_j\}$  and  $\{\mathcal{J}_j\}$  with  $\tilde{\mathcal{J}}_j$  a set of  $\tilde{d}_j$ -cylinders and  $\{\mathcal{J}_j\}$  a set of  $d_j$ -cylinders such that  $\tilde{\mathcal{J}}_0 = \mathcal{J}_0 = \{J_0\}$  and moreover:*

- (i) *For each  $j$ ,  $d_{j-1} < \tilde{d}_j \leq d_j$  and  $\mathcal{J}_j < \tilde{\mathcal{J}}_j < \mathcal{J}_{j-1}$ .*
- (ii) *For each  $\tilde{J}_j \in \tilde{\mathcal{J}}_j$  there exists a unique  $J_j \in \mathcal{J}_j$  such that  $J_j \subseteq \tilde{J}_j$ .*

We define the set  $\mathcal{Z}$  by

$$\mathcal{Z} = \bigcap_{j=0}^{\infty} \bigcup_{J_j \in \mathcal{J}_j} J_j = \bigcap_{j=0}^{\infty} \bigcup_{\tilde{J}_j \in \tilde{\mathcal{J}}_j} \tilde{J}_j.$$

We will refer to  $\mathcal{Z}$  as the set with pattern  $(\tilde{\mathcal{J}}_j, \mathcal{J}_j)$ .

### 4.1 The $\hat{\mu}$ -dimension of pattern sets. A Hungerford lemma.

Let  $\hat{\mu}$  be an atomless local weak Gibbs measure for the potential  $\phi : \Sigma_A^{\mathbb{Z}} \rightarrow \mathbb{R}$ . In this section we collect two results on the  $\hat{\mu}$ -dimension of a set  $\mathcal{Z}$  with pattern  $(\mathcal{J}_j, \tilde{\mathcal{J}}_j)$ .

We will assume that for all  $z \in \mathcal{Z}$  the constants  $c(z_n)$  in Remark 3.2 have an absolute upper bound in the times  $d_j$ , more precisely

$$\sup\{c(z_{d_j}) : z \in \mathcal{Z}, j \in \mathbb{N}\} \leq c < \infty \quad (27)$$

Notice also that since  $\mathcal{Z} \subset J_0$ , the set  $\mathcal{Z}$  is contained in a unique 0-cylinder, say  $C(0, z)$ . The importance of the above condition is that implies the following: consider the sequence  $\{\hat{s}_m = (c(z_0)c)^2 K_{m+1}^4\}$  then

$$1 \leq \hat{s}_m \leq \hat{s}_{m+1} \quad \text{and} \quad \lim_{m \rightarrow \infty} \frac{1}{m} \log \hat{s}_m = 0 \quad (28)$$

and

$$s_{n,m}(z) \leq \hat{s}_m \quad \text{for all } z \in \mathcal{Z} \quad \text{and for all } n = d_j < m$$

where  $s_{n,m}(z)$  is defined in Remark 3.2.

Notice that if  $\hat{\mu}$  is a weak Gibbs measure (not local) then the condition (27) always holds. Also, for  $\hat{\mu}$  being local if for example

$$\#\{\sigma^{d_j}(J_j) : J_j \in \mathcal{J}_j\} < \infty,$$

then we also have (27).

Hence, from (27) and Remark 3.2 we have that for all  $m > d_j$

$$\frac{1}{\hat{s}_m} \frac{\hat{\mu}(\sigma^{d_j}(C(m, z)))}{\hat{\mu}(\sigma^{d_j}(C(d_j, z)))} \leq \frac{\hat{\mu}(C(m, z))}{\hat{\mu}(C(d_j, z))} \leq \hat{s}_m \frac{\hat{\mu}(\sigma^{d_j}(C(m, z)))}{\hat{\mu}(\sigma^{d_j}(C(d_j, z)))} \quad (29)$$

**Theorem 4.1.** *Let  $\mathcal{Z}$  a set with pattern  $(\mathcal{J}_j, \tilde{\mathcal{J}}_j)$  verifying (27). Let  $\{\alpha_j\}$ ,  $\{\beta_j\}$ ,  $\{\gamma_j\}$ , and  $\{\delta_j\}$  be sequences of positive numbers such that:*

$$e^{-\alpha_j} \leq \frac{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \leq e^{-\beta_j}, \quad e^{-\gamma_j} \leq \frac{\hat{\mu}(\sigma^{d_{j-1}}(J_j))}{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j))} \quad (30)$$

$$\hat{\mu}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j \cap J_{j-1})) := \sum_{\tilde{\mathcal{J}}_j \in \tilde{\mathcal{J}}_j, \tilde{\mathcal{J}}_j \subset J_{j-1}} \hat{\mu}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j)) \geq \delta_j \hat{\mu}(\sigma^{d_{j-1}}(J_{j-1})), \quad (31)$$

and let assume that for all  $\varepsilon > 0$  there exists  $c > 0$  such that

$$\hat{s}_m \leq \frac{c}{\hat{\mu}(C(m, z))^\varepsilon} \quad \text{for all } d_{j-1} < m < \tilde{d}_j \quad \text{and } z \in \mathcal{Z}. \quad (32)$$

Then

$$\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq D^- := \liminf_{j \rightarrow \infty} \frac{(\beta_1 + \dots + \beta_j) - \log[1/(\delta_1 \dots \delta_{j+1})]}{(\alpha_1 + \dots + \alpha_j) + (\gamma_1 + \dots + \gamma_j) + \log[\hat{s}_{d_1} \dots \hat{s}_{d_{j-1}}]}$$

**Remark 4.1.** Notice that since  $\delta_j \leq 1$  and  $\beta_j \leq \alpha_j$ , then  $D^- \leq 1$ .

**Corollary 4.1.** If

$$\liminf_{j \rightarrow \infty} \frac{(\alpha_1 + \dots + \alpha_j) + (\gamma_1 + \dots + \gamma_j)}{\tilde{d}_j} > 0$$

then

$$D^- := \liminf_{j \rightarrow \infty} \frac{(\beta_1 + \dots + \beta_j) - \log[1/(\delta_1 \dots \delta_{j+1})]}{(\alpha_1 + \dots + \alpha_j) + (\gamma_1 + \dots + \gamma_j)}$$

If  $\hat{\mu}$  is a weak Gibbs measure we can eliminate  $\sigma^{d_{j-1}}$  from the conditions (30) and (31) and the conclusion of theorem (4.1) holds. Moreover, for  $\hat{\mu}$  a Gibbs measure the we can state the following corollary. The proofs are similar with the obvious changes.

**Corollary 4.2.** If  $\hat{\mu}$  is a Gibbs measure

$$e^{-\alpha_j} \leq \frac{\hat{\mu}(\tilde{\mathcal{J}}_j)}{\hat{\mu}(J_{j-1})} \leq e^{-\beta_j}, \quad e^{-\gamma_j} \leq \frac{\hat{\mu}(J_j)}{\hat{\mu}(\tilde{\mathcal{J}}_j)}, \quad \sum_{\tilde{\mathcal{J}}_j \in \tilde{\mathcal{J}}_j, \tilde{\mathcal{J}}_j \subset J_{j-1}} \hat{\mu}(\tilde{\mathcal{J}}_j) \geq \delta_j \hat{\mu}(J_{j-1}),$$

and

$$\liminf_{j \rightarrow \infty} [(\alpha_1 + \dots + \alpha_j) + (\gamma_1 + \dots + \gamma_j)]/j^\eta > 0 \quad \text{for some } \eta > 1. \quad (33)$$

Then

$$\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq \liminf_{j \rightarrow \infty} \frac{(\beta_1 + \dots + \beta_j) - \log[1/(\delta_1 \dots \delta_{j+1})]}{(\alpha_1 + \dots + \alpha_j) + (\gamma_1 + \dots + \gamma_j)}$$

**Remark 4.2.** Notice that if  $\tilde{\mathcal{J}}_j = \mathcal{J}_j$  i.e.  $\gamma_j = 0$  and  $\alpha_j = \alpha$ ,  $\beta_j = \beta$  and  $\delta_j = \delta$  for all  $j$ , then (33) holds and therefore  $\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq (\beta + \log \delta)/\alpha$ . This case corresponds to the classical Hungerford lemma.



*Proof of theorem 4.1.* We construct a probability measure  $\nu$  supported on  $\mathcal{Z}$  in the following way: We define  $\nu(J_0) = 1$  and for each set  $J_j \in \mathcal{J}_j$  we write

$$\nu(J_j) = \nu(\tilde{J}_j) = \frac{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j))}{\sum_{\tilde{J} \in \tilde{\mathcal{J}}_j, J \subset J_{j-1}} \hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}))} \nu(J_{j-1})$$

where  $J_{j-1}$  and  $\tilde{J}_j$  denote the unique sets in  $\mathcal{J}_{j-1}$  and  $\tilde{\mathcal{J}}_j$  respectively, such that  $J_j \subseteq \tilde{J}_j \subset J_{j-1}$ . As usual, for any cylinder  $B$ , the  $\nu$ -measure of  $B$  is defined by

$$\nu(B) = \nu(B \cap \mathcal{Z}) = \inf \sum_{U \in \mathcal{U}} \nu(U)$$

where the infimum is taken over all the coverings  $\mathcal{U}$  of  $B \cap \mathcal{Z}$  with sets in  $\bigcup \mathcal{J}_j$ .

We will show that for all  $\Lambda_1 < D^-$  there exists a positive constant  $c$  such that for all  $z \in \mathcal{Z}$  and  $m$  large enough,

$$\nu(C(m, z)) \leq c (\hat{\mu}(C(m, z)))^{\Lambda_1} \quad (34)$$

and therefore, for all covering  $\mathcal{U}$  we have that  $\sum_{U \in \mathcal{U}} (\hat{\mu}(U))^{\Lambda_1} > 0$  and we get the lower bound for the  $\hat{\mu}$ -dimension.

To prove (34) let us suppose first that  $C(m, z) = J_j$  for some  $J_j \in \mathcal{J}_j$ . From property (31) we have that

$$\nu(J_j) \leq \frac{1}{\delta_j \delta_{j-1} \cdots \delta_1} \frac{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j)) \cdots \hat{\mu}(\sigma^{d_0}(\tilde{J}_1))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1})) \cdots \hat{\mu}(\sigma^{d_0}(J_0))} \quad (35)$$

Then by (30)

$$\nu(J_j) \leq \frac{e^{-(\beta_j + \beta_{j-1} + \cdots + \beta_1)}}{\delta_j \delta_{j-1} \cdots \delta_1}. \quad (36)$$

Also we have from (29) and (30) that

$$\frac{\hat{\mu}(J_j)}{\hat{\mu}(J_{j-1})} \geq \frac{1}{\hat{s}_{d_j}} \frac{\hat{\mu}(\sigma^{d_{j-1}}(J_j))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \frac{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j))}{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j))} \geq \frac{1}{\hat{s}_{d_j}} e^{-(\alpha_j + \gamma_j)}$$

Wi fix  $\varepsilon > 0$  so that  $\frac{\Lambda_1 + \varepsilon}{1 - \varepsilon} < D^-$  and by using (32)

$$\hat{\mu}(J_j)^{1-\varepsilon} \geq \frac{e^{-[(\gamma_j + \alpha_j) + (\gamma_{j-1} + \alpha_{j-1}) + \cdots + (\gamma_1 + \alpha_1)]}}{\hat{s}_{d_{j-1}} \cdots \hat{s}_{d_1}} \hat{\mu}(J_0) \quad (37)$$

In order to get (34) for any  $C(m, z)$  with  $z \in \mathcal{Z}$ , we will get an stronger condition for the cylinders  $\tilde{J}_j$  and  $J_j$ , more precisely, for  $j$  large

$$\nu(\tilde{J}_j) = \nu(J_j) \leq c \delta_{j+1} \hat{\mu}(J_j)^{\Lambda_1 + \varepsilon} \quad (38)$$

for some positive constant  $c$ . Since  $0 < (\Lambda_1 + \varepsilon)/(1 - \varepsilon) < D^-$  the inequality (38) follows from (36) and (37).

Now, let us suppose that  $C(m, z) \neq J_j$  for all  $j$  and for all  $J_j \in \mathcal{J}_j$ , i.e  $d_j < m < d_{j+1}$ . Since  $z \in \mathcal{Z}$  there exist  $J_j \in \mathcal{J}_j$  and  $J_{j+1} \in \mathcal{J}_{j+1}$  such that  $J_{j+1} \subset C(m, z) \subset J_j$ .

If  $C(m, z) \subset \tilde{J}_{j+1}$ , then from the definition of  $\nu$  and (38) for  $\tilde{J}_{j+1}$  we get

$$\nu(C(m, z)) = \nu(\tilde{J}_{j+1}) = \nu(J_{j+1}) \leq c (\hat{\mu}(J_{j+1}))^{\Lambda_1} \leq c (\hat{\mu}(C(m, z)))^{\Lambda_1}.$$

Otherwise  $C(m, z)$  contains sets of the family  $\tilde{\mathcal{J}}_{j+1}$ , then  $d_j < m < \tilde{d}_{j+1}$  and we have that

$$\begin{aligned} \nu(C(m, z)) &= \sum_{\substack{\tilde{J}_{j+1} \in \tilde{\mathcal{J}}_{j+1} \\ \tilde{J}_{j+1} \subseteq C(m, z)}} \nu(J_{j+1}) = \sum_{\substack{\tilde{J}_{j+1} \in \tilde{\mathcal{J}}_{j+1} \\ \tilde{J}_{j+1} \subseteq C(m, z)}} \frac{\hat{\mu}(\sigma^{d_j}(\tilde{J}_{j+1}))}{\hat{\mu}(\sigma^{d_j}(\tilde{\mathcal{J}}_{j+1} \cap J_j))} \nu(J_j) \\ &= \frac{\nu(J_j)}{\hat{\mu}(\sigma^{d_j}(\tilde{\mathcal{J}}_{j+1} \cap J_j))} \sum_{\substack{\tilde{J}_{j+1} \in \tilde{\mathcal{J}}_{j+1} \\ \tilde{J}_{j+1} \subseteq C(m, z)}} \hat{\mu}(\sigma^{d_j}(\tilde{J}_{j+1})) \leq \frac{\nu(J_j)}{\hat{\mu}(\sigma^{d_j}(\tilde{\mathcal{J}}_{j+1} \cap J_j))} \hat{\mu}(\sigma^{d_j}(C(m, z))). \end{aligned} \quad (39)$$

Using property (31) we obtain that

$$\nu(C(m, z)) \leq \frac{\nu(J_j)}{\delta_{j+1} \hat{\mu}(\sigma^{d_j}(J_j))} \hat{\mu}(\sigma^{d_j}(C(m, z))). \quad (40)$$

And by using (29) and (32)

$$\nu(C(m, z)) \leq \frac{\hat{s}_m \nu(J_j)}{\delta_{j+1} \hat{\mu}(J_j)} \hat{\mu}(C(m, z)) \leq \frac{c \nu(J_j)}{\delta_{j+1} \hat{\mu}(J_j)} \hat{\mu}(C(m, z))^{1-\varepsilon}$$

Hence, by (38) we have that for some  $c' > 0$

$$\nu(C(m, z)) \leq c' \frac{1}{\hat{\mu}(J_j)^{1-\Lambda_1-\varepsilon}} \hat{\mu}(C(m, z))^{1-\varepsilon}$$

and since  $\hat{\mu}(J_j) \geq \hat{\mu}(C(m, z))$  we get (34).  $\square$

The last result in this section is also a kind of Hungerford lemma. The motivation is the following: for some  $\mathcal{Z}$  we will be able to construct (by using thermodynamic formalism) an alternative measure  $\hat{\mathbf{m}}$  such that  $\hat{\mathbf{m}}(\tilde{J}_j) \leq c \hat{\mu}(J_j)^t$ , and we can use  $\hat{\mathbf{m}}$  (instead of  $\hat{\mu}$ ) to equidistribute the mass in the standard measure with support in  $\mathcal{Z}$ . The definition of  $t$  in the next theorem is the way of expressing the above relation between  $\hat{\mathbf{m}}$  and  $\hat{\mu}$  in the case of  $\hat{\mu}$  being a local weak Gibbs measure.

**Theorem 4.2.** *Let  $\{\gamma_j\}$  be sequence of positive numbers such that*

$$\frac{\hat{\mu}(\sigma^{d_{j-1}}(J_j))}{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j))} \geq e^{-\gamma_j}. \quad (41)$$

*Let us suppose that there exists a finite measure  $\mathbf{m}$  on  $\Sigma_A^{\mathcal{I}}$  such that*

$$\hat{\mathbf{m}}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j \cap J_{j-1})) := \sum_{\tilde{J}_j \in \tilde{\mathcal{J}}_j, \tilde{J}_j \subset J_{j-1}} \hat{\mathbf{m}}(\sigma^{d_{j-1}}(\tilde{J}_j)) \geq \bar{\delta}_j \hat{\mathbf{m}}(\sigma^{d_{j-1}}(J_{j-1})), \quad (42)$$

*for some sequence  $\{\bar{\delta}_j\}$  with  $0 < \bar{\delta}_j \leq 1$ . And moreover, for some  $0 < t \leq 1$ , if  $C(m, z) \subset J_{j-1}$  with  $z \in \mathcal{Z}$  then*

$$\frac{\hat{\mathbf{m}}(\sigma^{d_{j-1}}(C(m, z)))}{\hat{\mathbf{m}}(\sigma^{d_{j-1}}(J_{j-1}))} \leq c \bar{\delta}_j \left( \eta_m \frac{\hat{\mu}(\sigma^{d_{j-1}}(C(m, z)))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \right)^t \quad \text{for } d_{j-1} \leq m \leq \tilde{d}_j \quad (43)$$

*with  $c$  a positive constant and*

$$\eta_m = \begin{cases} e^{-\bar{\gamma}_j / \hat{s}_{d_j}} & \text{for } m = \tilde{d}_j \\ 1 / \hat{s}_m & \text{for } d_{j-1} \leq m < \tilde{d}_j \end{cases} \quad (44)$$

*Then*

$$\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq t.$$

If  $\hat{\mu}$  is weak Gibbs we can delete  $\sigma^{d_{j-1}}$  from (41), (42) and (54) and the conclusion of theorem 4.2 holds. For  $\hat{\mu}$  Gibbs we have the following corollary. Again the proofs are similar with the obvious changes.

**Corollary 4.3.** *Let  $\hat{\mu}$  be a Gibbs measure such that*

$$\frac{\hat{\mu}(J_j)}{\hat{\mu}(\tilde{J}_j)} \geq e^{-\gamma_j}.$$

*Let us suppose that, there is a  $\sigma$ -Gibbs measure  $\hat{\mathbf{m}}$  for the potential  $t\phi$ , for some  $0 \leq t \leq 1$ ,*

$$\sum_{\tilde{J}_j \in \tilde{\mathcal{J}}_j, \tilde{J}_j \subset J_{j-1}} \hat{\mathbf{m}}(\tilde{J}_j) \geq \bar{\delta} \hat{\mathbf{m}}(J_{j-1}) \quad \text{for some constant } \bar{\delta} > 0,$$

*and*

$$\limsup_{j \rightarrow \infty} \frac{\gamma_j}{d_j - d_{j-1}} < \frac{1}{t} [P_G(t\phi) - tP_G(\phi)] \quad (45)$$

*Then*

$$\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq t.$$

**Remark 4.3.** *Recall that if  $\Sigma_A^{\mathcal{I}}$  has the BIP property (in particular, if the alphabet is finite) and the potential  $\phi$  satisfies the Walter condition (this condition is weaker than Hölder), has finite Gurevich pressure and  $V_1(\phi) < \infty$ , then for all  $0 < t \leq 1$ , the RPF measure for  $t\phi$  is a  $\sigma$ -Gibbs measure*

**Remark 4.4.** *Notice that from (45) follows that  $\hat{\mathbf{m}}(\tilde{J}_j) \leq C \hat{\mu}(J_j)^t$ .*

*Proof of theorem 4.2.* We proceed as in the proof of theorem 4.1 but we define the measure  $\nu$  by:  $\nu(J_0) = 1$  and for each set  $J_j \in \mathcal{J}_j$  we write

$$\nu(J_j) = \nu(\tilde{J}_j) = \frac{\hat{\mathbf{m}}(\sigma^{d_{j-1}}(\tilde{J}_j))}{\hat{\mathbf{m}}(\sigma^{d_{j-1}}(\tilde{\mathcal{J}}_j \cap J_{j-1}))} \nu(J_{j-1})$$

where  $J_{j-1}$  and  $\tilde{J}_j$  denote the unique sets in  $\mathcal{J}_{j-1}$  and  $\tilde{\mathcal{J}}_j$  respectively, such that  $J_j \subseteq \tilde{J}_j \subset J_{j-1}$ . From (42) and (44) we get the following estimate instead of (35)

$$\nu(J_j) \leq \left( \frac{e^{-\bar{\gamma}_j}}{\hat{s}_{d_j}} \frac{\hat{\mu}(\sigma^{d_{j-1}}(\tilde{J}_j))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \cdots \frac{e^{-\bar{\gamma}_1}}{\hat{s}_{d_1}} \frac{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))}{\hat{\mu}(\sigma^{d_0}(J_0))} \right)^t$$

and therefore for some  $c' > 0$

$$\nu(J_j) \leq \left( \frac{1}{\hat{s}_{d_j}} \frac{\hat{\mu}(\sigma^{d_{j-1}}(J_j))}{\hat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \cdots \frac{1}{\hat{s}_{d_1}} \frac{\hat{\mu}(\sigma^{d_0}(J_1))}{\hat{\mu}(\sigma^{d_0}(J_0))} \right)^t \leq \left( \frac{\hat{\mu}(J_j)}{\hat{\mu}(J_{j-1})} \cdots \frac{\hat{\mu}(J_1)}{\hat{\mu}(J_0)} \right)^t \leq c' \hat{\mu}(J_j)^t$$

Also, instead of (40) for  $d_j < m < \tilde{d}_{j+1}$  we have

$$\nu(C(m, z)) \leq \frac{\nu(J_j)}{\delta_{j+1} \hat{\mathbf{m}}(\sigma^{d_j}(J_j))} \hat{\mathbf{m}}(\sigma^{d_j}(C(m, z)))$$

Hence by using (44)

$$\nu(C(m, z)) \leq c \left( \frac{1}{\hat{s}_m} \frac{\hat{\mu}(\sigma^{d_j}(C(m, z)))}{\hat{\mu}(\sigma^{d_j}(J_j))} \right)^t \nu(J_j) \leq c'' \left( \frac{\hat{\mu}(C(m, z))}{\hat{\mu}(J_j)} \right)^t \hat{\mu}(J_j)^t = c'' \hat{\mu}(C(m, z))^t$$

The rest of the proof is similar. □

## 5 Target-ball set for the shift transformation

Let  $(\Sigma_A^{\mathcal{I}}, d, \sigma, \hat{\mu})$  be a topologically mixing topological Markov chain endowed with an atomless local weak  $\sigma$ -Gibbs measure  $\hat{\mu}$ . In the case of infinite countable alphabet we will assume that the potential  $\phi$  has summable variations and so the constant  $P < \infty$  is the Gurevich pressure  $P_G(\phi)$ . Given a point  $w \in \Sigma_A^{\mathcal{I}}$ , a  $N$ -cylinder  $\hat{P}$ , and a sequence  $\{\ell_n\} \subset \mathbb{N}$  we want to estimate the size of the set

$$W_\sigma(\hat{P}, \ell_n, w) = \{z \in \hat{P} : \sigma^k(z) \in C(\ell_k, w) \text{ for infinitely many } k\}.$$

In some subsections some extra properties on  $\phi$  and  $\{\ell_n\}$  are required, we will indicate them at the beginning of each subsection.

Recall that if  $\hat{\mu}$  is ergodic,  $\phi \in L^1(\hat{\mu})$  and  $\sum_{n \geq 1} V_n(\phi) < \infty$ , then (see (19)) for  $w$   $\hat{\mu}$ -a.e

$$0 \geq \lim_{n \rightarrow \infty} \frac{1}{n} \log \hat{\mu}(C(n, w)) = -P + \int \phi d\hat{\mu} \quad (46)$$

Therefore if  $P - \int \phi d\hat{\mu} > 0$  and  $\underline{\nu} := \liminf_{n \rightarrow \infty} \ell_n/n > 0$ , given  $\varepsilon > 0$  small enough we have that

$$\sum_n \hat{\mu}(C(\ell_n, w)) \leq c \sum_n e^{-n(\underline{\nu} - \varepsilon)[P - \int \phi d\hat{\mu} - \varepsilon]} < \infty$$

and so, under these hypothesis, the Borel-Cantelli lemma implies that  $W_\sigma(\hat{P}, \ell_n, w)$  has zero  $\hat{\mu}$ -measure for  $w$   $\hat{\mu}$  a.e.

Later, in order to get a lower bound for the  $\hat{\mu}$ -dimension of  $W_\sigma(\hat{P}, \ell_n, w)$  we will require the following property on the center  $w$  of the target.

**Definition 5.1.** A point  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathcal{I}}$  is a  $\hat{\mu}$ -hitting point if there exists an increasing sequence  $\mathcal{I}(w) = \{p_j\} \subset \mathbb{N}$  such that the constants  $c(w_n)$  in Remark 3.2 have an absolute upper bound in the times  $p_j$ , more precisely

$$\sup\{c(w_{p_j}) : j \in \mathbb{N}\} \leq c < \infty$$

**Remark 5.1.** If the alphabet is finite or  $\hat{\mu}$  is a (not local) weak Gibbs measure, then any point  $w$  is a  $\hat{\mu}$ -hitting point. In any case, by the recurrence theorem of Poincaré we know that for  $w = (w_0, w_1, \dots)$   $\hat{\mu}$ -a.e there exists an increasing sequence  $\{p_j\} \subset \mathbb{N}$  such that  $w_{p_j} = w_0$  and so the set of  $\hat{\mu}$ -hitting point has full  $\hat{\mu}$ -measure. In fact, if that recurrence sequence exists, then  $w$  is a hitting point for any local weak Gibbs measure.

The importance of the above definition is indicated in the following lemma.

**Lemma 5.1.** Let  $C$  be a 0-cylinder and  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathcal{I}}$  be a  $\hat{\mu}$ -hitting point with sequence  $\mathcal{I}(w) = \{p_j\}$ . There exists a sequence  $\{\hat{s}_m\}$  with

$$1 \leq \hat{s}_m \leq \hat{s}_{m+1} \quad \text{and} \quad \lim_{m \rightarrow \infty} \frac{1}{m} \log \hat{s}_m = 0$$

such that if  $C(n, z) \subset C$  with  $\sigma^n(C(n, z)) = C(0, \sigma^{p_j}(w))$  for some  $j$  then

$$\frac{1}{\hat{s}_m} \frac{\hat{\mu}(\sigma^n(C(m, z)))}{\hat{\mu}(\sigma^n(C(n, z)))} \leq \frac{\hat{\mu}(C(m, z))}{\hat{\mu}(C(n, z))} \leq \hat{s}_m \frac{\hat{\mu}(\sigma^n(C(m, z)))}{\hat{\mu}(\sigma^n(C(n, z)))} \quad \text{for all } m > n$$

and for  $m = n - 1$

$$\frac{1}{\hat{s}_{n-1}} \frac{1}{\hat{\mu}(\sigma^n(C(n, z)))} \leq \frac{\hat{\mu}(C(n-1, z))}{\hat{\mu}(C(n, z))} \leq \hat{s}_{n-1} \frac{1}{\hat{\mu}(\sigma^n(C(n, z)))}$$

Moreover, if  $\hat{\mu}$  is a local  $\sigma$ -Gibbs measure (not weak), then the sequence  $\{\hat{s}_m\}$  is constant.

*Proof.* Since  $C(n, z) \subset C$  we have that  $z_0$  is constant (is the symbol of  $C$ ) and since  $\sigma^n(C(n, z)) = C(0, \sigma^{p_j}(w))$  we have that  $z_n = w_{p_j}$ . Therefore the constant  $c(z_n)$  in Remark 3.2 is equal to  $c(w_{p_j})$ , and  $c(w_{p_j}) \leq c < \infty$  by definition of  $\hat{\mu}$ -hitting point. We can take  $\hat{s}_m = (c(z_0)c)^2 K_{m+1}^4$ , then the result follows from Remark 3.2.  $\square$

## 5.1 Estimation of the $\hat{\mu}$ -dimension

First, we will give an upper bound for the  $\hat{\mu}$ -dimension of  $W_\sigma(\hat{P}, \ell_n, w)$ .

**Proposition 5.1.** *Let*

$$\underline{s} := \liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\hat{\mu}(C(\ell_n, w))} > 0 \quad (47)$$

*Then,*

$$\text{Dim}_{\hat{\mu}} W_\sigma(\hat{P}, \ell_n, w) \leq \inf\{t > 0 : P_G(t\phi) - P_G(\phi)t < \underline{s}t\},$$

*Proof.* For each  $N \in \mathbb{N}$  we have the following covering of  $W_\sigma(\hat{P}, \ell_n, w)$ :

$$\bigcup_{n=N}^{\infty} \{C(n + \ell_n, z) : \sigma^n(z) = w, z_0 = i\}.$$

with  $i$  such that  $\hat{P} \subset C_i$ . Moreover, given  $\varepsilon > 0$  and  $t > 0$ , from (47), remark 3.2 and since  $\sigma^n(z) = w$  we have for  $n$  large enough

$$\frac{\hat{\mu}(C(n + \ell_n, z))}{\hat{\mu}(C(n, z))} \leq s_{n, n+\ell_n} \frac{\hat{\mu}(C(\ell_n, w))}{\hat{\mu}(C(0, w))} \leq \frac{c(i)^2}{c(w_0)^2 \hat{\mu}(C(0, w))} e^{-n(\underline{s} - \varepsilon/(2t))} \quad (48)$$

Therefore for  $t > 0$  and  $N$  large

$$\sum_{n \geq N} \sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n + \ell_n, z))^t \leq c \sum_{n \geq N} e^{-n(\underline{s}t - \varepsilon/2)} \sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n, z))^t$$

with  $c$  a positive constant. But for  $n$  large

$$\hat{\mu}(C(n, z))^t \leq c(i)^{2t} \exp[-(n+1)(tP_G(\phi) - \varepsilon/4) + t \sum_{j=0}^n \phi \circ \sigma^j(z)]$$

and so

$$\sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n, z))^t \leq c(i)^{2t} e^{-(n+1)(tP_G(\phi) - \varepsilon/4)} \sum_{\sigma^n(z)=w, z_0=i} e^{\sum_{j=0}^n (t\phi \circ \sigma^j(z))}$$

Since  $\Sigma_A^T$  is topologically mixing there exists  $u = (u_0, u_1, \dots, u_{k_0}) \in \Sigma_A^T|_{k_0}$  (for some  $k_0$ ) with  $u_0 = w_0$  and  $u_{k_0} = i$ , and therefore by using that  $\phi$  has summable variations we have that

$$\sum_{\sigma^n(z)=w, z_0=i} e^{\sum_{j=0}^n (t\phi \circ \sigma^j(z))} \leq c \sum_{\sigma^{n+k_0}(v)=v, v_0=i} e^{\sum_{j=0}^{n+k_0-1} (t\phi \circ \sigma^j(v))} = c Z_{n+k_0}(t\phi, i)$$

for some constant  $c > 0$ . As  $P_G(t\phi) < \infty$ , for  $n$  large,

$$\sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n, z))^t \leq c c(i)^{2t} e^{-(n+1)(tP_G(\phi) - \varepsilon/4)} e^{(n+k_0)(P_G(t\phi) + \varepsilon/4)} = c' e^{n(P_G(t\phi) - tP_G(\phi) + \varepsilon/2)}$$

Therefore for  $N$  large

$$\sum_{n \geq N} \sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n + \ell_n, z))^t \leq c'' \sum_{n \geq N} e^{-n(\underline{s}t - P_G(t\phi) + tP_G(\phi) - \varepsilon)}$$

for some constant  $c'' > 0$ . Therefore, for  $\varepsilon > 0$  and  $t > 0$  such that  $\underline{s}t - P_G(t\phi) + tP_G(\phi) - \varepsilon > 0$  we have that

$$\sum_{n \geq N} \sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n + \ell_n, z))^t \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

and so  $M_{\mathcal{F}, \hat{\mu}^t}(W_\sigma(\hat{P}, \ell_n, w)) = 0$ . □

**Remark 5.2.** *In the case  $\underline{s} = \infty$ , the same argument (with the obvious minor changes) gives that  $\text{Dim}_{\hat{\mu}} W_\sigma(\hat{P}, \ell_n, w) = 0$ , as it is implicitly stated in the proposition.*

### 5.1.1 Lower bound by using some related weak Gibbs measures

In order to get the lower bound for the  $\hat{\mu}$ -dimension, we will need the existence of weak Gibbs measures for some related new potentials, more precisely for  $t\phi$ . In this section we will also require that  $V_1(\phi) < \infty$  for the potential  $\phi$  of the measure  $\hat{\mu}$ , and if  $\hat{\mu}$  is weak (i.e  $\sup_n K_n = \infty$ ) then we need the sequence  $\{\ell_n\}$  verifies  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ .

The following lemma is a straightforward consequence of the definition of local weak Gibbs measures.

**Lemma 5.2.** *Let  $t > 0$  and suppose  $\hat{\mathbf{m}}$  is a weak Gibbs measure for the potential  $t\phi$ . Then there exist a sequence  $\{\hat{k}_l\}$  with*

$$1 \leq \hat{k}_l \leq \hat{k}_{l+1} \quad \text{and} \quad \lim_{l \rightarrow \infty} \frac{1}{l} \log \hat{k}_l = 0$$

such that for any  $l$ -cylinder  $C(l, z)$

$$\frac{\hat{\mathbf{m}}(C(l, z))}{\hat{\mathbf{m}}(C(0, z))} \leq \hat{c}(z_0)^2 \hat{k}_l e^{-l(P_G(t\phi) - tP_G(\phi))} \left( \frac{\hat{\mu}(C(l, z))}{\hat{\mu}(C(0, z))} \right)^t$$

Here  $\hat{c}(z_0) = c_{\hat{\mu}}(z_0)^t c_{\hat{\mathbf{m}}}(z_0)$  where  $c_{\hat{\mu}}(z_0)$  and  $c_{\hat{\mathbf{m}}}(z_0)$  denote the localness constants of the measures  $\hat{\mu}$  and  $\hat{\mathbf{m}}$ .

*Proof.* Since  $\phi$  has summable variation, the constant  $P$  for the local weak Gibbs measure  $\hat{\mu}$  is the Gurevich pressure  $P_G(\phi)$ . The potential of  $\hat{\mathbf{m}}$  has also summable variation since  $\phi$  has summable variation. So the constant  $P$  for  $\hat{\mathbf{m}}$  is the Gurevich pressure  $P_G(t\phi)$ . Let us denote by  $\{K_{n, \hat{\mathbf{m}}}\}$  and  $\{K_{n, \hat{\mu}}\}$  the sequences in Definition 3.1 for  $\hat{\mathbf{m}}$  and  $\hat{\mu}$  respectively. Then for some  $x$

$$\frac{\hat{\mathbf{m}}(C(l, x))}{\hat{\mathbf{m}}(C(0, x))} \leq c_{\hat{\mathbf{m}}}(z_0)^2 K_{1, \hat{\mathbf{m}}} K_{l+1, \hat{\mathbf{m}}} e^{-lP_G(t\phi)} e^{t \sum_{j=1}^l \phi \circ \sigma^j(x)}$$

and for some  $z \in C(l, x)$

$$\left( \frac{\hat{\mu}(C(l, z))}{\hat{\mu}(C(0, z))} \right)^t \geq \left( \frac{1}{c_{\hat{\mu}}(z_0)^2 K_{1, \hat{\mu}} K_{l+1, \hat{\mu}}} \right)^t e^{-ltP_G(\phi)} e^{t \sum_{j=1}^l \phi \circ \sigma^j(z)}$$

We get the result by using that  $\sum_{n=1}^{\infty} V_n(\phi) < \infty$ . □

Later we will need the following result, the proof is similar to the previous one.

**Lemma 5.3.** *Let  $\gamma \geq 0$ ,  $t > 0$  and suppose  $\hat{\mathbf{m}}$  is a weak Gibbs measure for the potential  $t\phi$ . Then there exist a sequence  $\{\hat{k}_l\}$  with*

$$1 \leq \hat{k}_l \leq \hat{k}_{l+1} \quad \text{and} \quad \lim_{l \rightarrow \infty} \frac{1}{l} \log \hat{k}_l = 0$$

such that for any  $l$ -cylinder  $C(l, z)$

$$\frac{\hat{\mathbf{m}}(C(l+1, z))}{\hat{\mathbf{m}}(C(l, z))^{1+\gamma}} \leq \hat{c}(z_0)^{2+\gamma} \hat{k}_{l+1} e^{((l+1)\gamma-1)(P_G(t\phi) - tP_G(\phi))} \left( \frac{\hat{\mu}(C(l+1, z))}{\hat{\mu}(C(l, z))^{1+\gamma}} \right)^t$$

**Theorem 5.1.** *Let  $w$  be a  $\hat{\mu}$ -hitting point with sequence  $\mathcal{I}(w) = \{p_j\}$  and let suppose*

$$\bar{s} := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\hat{\mu}(C(\ell_n, w))} < \infty, \quad (49)$$

there is a positive number  $t$  such that

$$\infty > P_G(t\phi) - tP_G(\phi) \geq t\bar{s} + \varepsilon \quad \text{for some } \varepsilon > 0 \quad (50)$$



and moreover, there exists a mixing local weak  $\sigma$ -Gibbs measure  $\widehat{\mathbf{m}}$  for the potential  $t\phi$  with local constants such that

$$\sup\{c_{\widehat{\mathbf{m}}}(w_{p_j}) : j \in \mathbb{N}\} < \infty, \quad (51)$$

and  $\phi \in L^1(\widehat{\mathbf{m}})$ .

Then, there exists a  $(\widetilde{\mathcal{J}}_j, \mathcal{J}_j)$  pattern set  $\mathcal{Z}$ , with  $\widetilde{\mathcal{J}}_j$  a set of  $\widetilde{d}_j$ -cylinders and  $\mathcal{J}_j$  a set of  $d_j$ -cylinders, such that

$$\mathcal{Z} \subset \{z \in J_0 : \sigma^{\widetilde{d}_j}(z) \in C(\ell_{\widetilde{d}_j}, w) \text{ for } j \in \mathbb{N} \setminus \{0\}\} \subset W_\sigma(\widehat{P}, \ell_n, w),$$

and verifying:

(i)

$$e^{-\gamma_j} \leq \frac{\widehat{\mu}(\sigma^{d_{j-1}}(J_j))}{\widehat{\mu}(\sigma^{d_{j-1}}(\widetilde{J}_j))} \quad \text{with} \quad \gamma_j = \widetilde{d}_j(\overline{s} + \varepsilon/(2t)) \quad (52)$$

(ii) There exists  $\overline{\delta} > 0$  such that

$$\widehat{\mathbf{m}}(\sigma^{d_{j-1}}(\widetilde{\mathcal{J}}_j \cap J_{j-1})) := \sum_{\widetilde{J}_j \in \widetilde{\mathcal{J}}_j, \widetilde{J}_j \subset J_{j-1}} \widehat{\mathbf{m}}(\sigma^{d_{j-1}}(\widetilde{J}_j)) \geq \overline{\delta} \widehat{\mathbf{m}}(\sigma^{d_{j-1}}(J_{j-1})), \quad (53)$$

(iii) For  $C(m, z) \subset J_{j-1}$

$$\frac{\widehat{\mathbf{m}}(\sigma^{d_{j-1}}(C(m, z)))}{\widehat{\mathbf{m}}(\sigma^{d_{j-1}}(J_{j-1}))} \leq c \overline{\delta} \left( \eta_m \frac{\widehat{\mu}(\sigma^{d_{j-1}}(C(m, z)))}{\widehat{\mu}(\sigma^{d_{j-1}}(J_{j-1}))} \right)^t \quad \text{for } d_{j-1} \leq m \leq \widetilde{d}_j \quad (54)$$

with  $c > 0$  a constant and

$$\eta_m = \begin{cases} e^{-\overline{\gamma}_j / \widehat{s}_{d_j}} & \text{for } m = \widetilde{d}_j \\ 1 / \widehat{s}_m & \text{for } d_{j-1} \leq m < \widetilde{d}_j \end{cases} \quad (55)$$

Here  $\overline{\delta}$  is the constant in (ii) and  $\widehat{s}_m$  is given by lemma 5.1.

The next corollary follows from above result and theorem 4.2

**Corollary 5.1.**

$$\text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \geq T^-,$$

with  $T^-$  the supremum of the set of  $0 < t \leq 1$  such that

$$\infty > P_G(t\phi) - P_G(\phi)t > \overline{s}t,$$

and there exists a mixing local weak  $\sigma$ -Gibbs measure  $\widehat{\mathbf{m}}$  for the potential  $t\phi$  with local constants verifying (51), and  $\phi \in L^1(\widehat{\mathbf{m}})$ .

**Remark 5.3.** Notice that the convexity of the Gurevich pressure (see section 3) imply the convexity of the function  $G : [0, 1] \rightarrow [0, +\infty]$  defined as  $G(t) := P_G(t\phi) - P_G(\phi)t$ , and since  $G(1) = 0$  we have that  $G(t)$  is decreasing. Hence, we have that for  $\underline{s} > 0$

$$\sup\{t > 0 : \infty > G(t) > \overline{s}t\} \leq \inf\{t > 0 : G(t) < \underline{s}t\}$$

We recall that  $\text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \leq \inf\{t > 0 : G(t) < \underline{s}t\}$ .

Later in section 6 we will use the following remark to get our Hausdorff dimension results for Markov transformations with infinite countable alphabet. This result is not necessary in the study of the  $\widehat{\mu}$ -dimension of the target-ball set for the shift transformation. The proof is given in section 5.1.3.

**Remark 5.4.** Under the hypothesis of theorem 5.1 let us also suppose that the center of the target  $w$  verifies

$$\underline{s} = \liminf_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\hat{\mu}(C(\ell_n, w))} > 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \frac{\hat{\mu}(C(n+1, w))}{\hat{\mu}(C(n, w))} > 0.$$

If  $P_G(\phi) - \int \phi d\hat{\mathbf{m}} > 0$  and the collection  $\mathcal{P} = \{C(0, \sigma^{p_j}(w)) : p_j \in \mathcal{I}(w)\}$  is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good for all  $\varepsilon$  small enough, then there exists a  $(\tilde{\mathcal{J}}_j, \mathcal{J}_j)$  pattern set  $\mathcal{Z}_\varepsilon \subset W_\sigma(\hat{P}, \ell_n, w)$  verifying the conditions in theorem 5.1 and with the following extra property:

For all

$$\gamma \geq \frac{3\varepsilon}{tP_G(\phi) - t \int \phi d\hat{\mathbf{m}} - \varepsilon} > 0$$

there is a constant  $c > 0$  such that for all  $z \in \mathcal{Z}$

$$\frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \geq c \quad \text{for} \quad d_j \leq n < \tilde{d}_{j+1}$$

By construction (also in theorem 5.1) we have that

$$\sigma^{\tilde{d}_{j+1}}(C(n, z)) = C(n - \tilde{d}_{j+1}, w) \quad \text{for} \quad \tilde{d}_{j+1} \leq n \leq d_{j+1}$$

Recall that if the collection  $\#\mathcal{P}$  is finite then we always have that  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good (since  $\hat{\mathbf{m}}$  is mixing). For  $\#\mathcal{P}$  infinite we know by corollary 3.1 that if  $\hat{\mathbf{m}}$  has summable uniform rate of mixing of order 3 in  $\mathcal{P}$  and  $\phi \in L^2(\hat{\mathbf{m}})$  then  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good.

*Proof of theorem 5.1.* Since  $\Sigma_A^{\mathcal{I}}$  is topologically mixing, for  $\tilde{d}_0$  large enough  $\sigma^{-\tilde{d}_0}(C(0, w)) \cap \hat{P} \neq \emptyset$ . Therefore, there exists a  $\tilde{d}_0$ -cylinder  $\tilde{J}_0 \subset \hat{P}$  such that  $\sigma^{\tilde{d}_0}(\tilde{J}_0) = C(0, w)$ . We define  $J_0 := \tilde{J}_0$  and  $\tilde{\mathcal{J}}_0 = \{\tilde{J}_0\}$ ,  $\mathcal{J}_0 = \{J_0\}$ .

To construct the family  $\tilde{\mathcal{J}}_1$  we will apply proposition 3.1 for the measure  $\hat{\mathbf{m}}$  with  $P_1 = P_2 = C(0, w)$ . We take  $N_1$  be a natural number large enough so that  $\mathcal{S}_N(M, \varepsilon)$  with  $N = M = N_1$  satisfies (20). Hereafter, for all  $n$ -cylinder  $C$ , we will denote by  $G_C^{-1}$  the composition of the  $n$  branches of  $\sigma^{-1}$  such that  $C = G_C^{-1}(\sigma^n(C))$ . We define  $\tilde{\mathcal{J}}_1$  as the family of cylinders  $G_{J_0}^{-1}(S)$  with  $S \in \mathcal{S}_{N_1}(N_1, \varepsilon)$ . Notice that by proposition 3.1

$$\hat{\mathbf{m}}(\sigma^{d_0}(\tilde{\mathcal{J}}_1 \cap J_0)) := \sum_{\tilde{J}_1 \in \tilde{\mathcal{J}}_1, \tilde{J}_1 \subset J_0} \hat{\mathbf{m}}(\sigma^{d_0}(\tilde{J}_1)) \geq \bar{\delta} \hat{\mathbf{m}}(\sigma^{d_0}(J_0)) \quad \text{with} \quad \bar{\delta} = \frac{1}{2} \hat{\mathbf{m}}(C(0, w)).$$

Let  $\mathcal{I}(w)$  be the sequence given by the definition of  $\hat{\mu}$ -hitting point. We choose  $N_1$  so that there exists  $p_i \in \mathcal{I}(w)$  for  $\tilde{d}_1 := d_0 + N_1$  such that

$$\ell_{\tilde{d}_1} \leq p_i < \ell_{\tilde{d}_1+1}. \quad (56)$$

We denote this natural number  $p_i$  by  $p(\tilde{d}_1)$ .

For each set  $S$  in  $\mathcal{S}_{N_1}(N_1, \varepsilon)$  we have that  $\sigma^{N_1}(S) = C(0, w)$ , then we take in each  $S$  the subset  $L := G_S^{-1}(C(p(\tilde{d}_1), w))$  and we denote by  $\mathcal{L}_1$  this family of sets. We define the family  $\mathcal{J}_1$  as the collection  $G_{J_0}^{-1}(L)$  with  $L \in \mathcal{L}_1$ .

Notice that by construction for all  $J_1 \in \mathcal{J}_1$  there exists a unique  $\tilde{J}_1 \in \tilde{\mathcal{J}}_1$  such that  $J_1 \subset \tilde{J}_1$ ,

$$\sigma^{\tilde{d}_1}(\tilde{J}_1) = C(0, w), \quad \sigma^{\tilde{d}_1}(J_1) = C(p(\tilde{d}_1), w) \subset C(\ell_{\tilde{d}_1}, w), \quad \sigma^{d_1}(J_1) = C(0, \sigma^{p(\tilde{d}_1)}(w)),$$

with  $\tilde{d}_1 := d_0 + N_1$  and  $d_1 := \tilde{d}_1 + p(\tilde{d}_1)$ .

Since  $\sigma^{d_0}(\tilde{J}_1) = S$  for some  $S \subset C(0, w)$  with  $\sigma^{N_1}(S) = C(0, w)$ , and  $\sigma^{d_0}(J_1) = L \subset S$  for some  $L$  with  $\sigma^{N_1}(L) = C(p(\tilde{d}_1), w)$ , then we have from lemma 5.1 (we may assume that  $0 \in \mathcal{I}(w)$ ) and (56) that

$$\frac{\hat{\mu}(\sigma^{d_0}(J_1))}{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))} = \frac{\hat{\mu}(L)}{\hat{\mu}(S)} \geq \frac{1}{\hat{s}_{N_1+p(\tilde{d}_1)}} \frac{\hat{\mu}(C(p(\tilde{d}_1), w))}{\hat{\mu}(C(0, w))} \geq \frac{1}{\hat{s}_{N_1+\ell_{\tilde{d}_1+1}}} \hat{\mu}(C(\ell_{\tilde{d}_1+1}, w)) \quad (57)$$

From (49) we have for  $N_1$  large enough that

$$\hat{\mu}(C(\ell_{\tilde{d}_1+1}, w)) \geq e^{-\tilde{d}_1(\bar{s}+\varepsilon/t)}$$

Recall that  $\lim_{m \rightarrow \infty} (\log \hat{s}_m)/m = 0$  with  $\{\hat{s}_m\}$  a constant sequence if  $\hat{\mu}$  is not weak (i.e  $\sup_n K_n < \infty$ ), and if  $\hat{\mu}$  is weak (i.e  $\sup_n K_n = \infty$ ) we are assuming that  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ . So, for  $N_1$  large enough we also have

$$\frac{1}{\hat{s}_{N_1+\ell_{\tilde{d}_1+1}}} \geq e^{-\varepsilon \tilde{d}_1/(2t)}$$

Hence

$$\frac{\hat{\mu}(\sigma^{d_0}(J_1))}{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))} \geq e^{-\gamma_1} \quad \text{with} \quad \gamma_1 = \tilde{d}_1(\bar{s} + \varepsilon/(2t))$$

From lemma 5.2 for  $d_0 \leq m \leq \tilde{d}_1$

$$\frac{\hat{\mathbf{m}}(\sigma^{d_0}(C(m, z)))}{\hat{\mathbf{m}}(\sigma^{d_0}(J_0))} \leq \hat{c}(w_0)^2 \hat{k}_{m-d_0} e^{-(m-d_0)[P_G(t\phi)-tP_G(\phi)]} \left( \frac{\hat{\mu}(\sigma^{d_0}(C(m, z)))}{\hat{\mu}(\sigma^{d_0}(J_0))} \right)^t$$

By taking  $d_0$  large enough we have for  $m \geq d_0$  that

$$\hat{c}(w_0)^2 \hat{k}_m \hat{s}_m^t \leq \bar{\delta} e^{m\varepsilon/2}$$

and therefore (we recall that  $\hat{k}_{m-d_0} \leq k_m$ ) by using (50) we get for some  $c > 0$

$$\begin{aligned} \frac{\hat{\mathbf{m}}(\sigma^{d_0}(C(m, z)))}{\hat{\mathbf{m}}(\sigma^{d_0}(J_0))} &\leq e^{d_0[P_G(t\phi)-tP_G(\phi)]} e^{-m[P_G(t\phi)-tP_G(\phi)-\varepsilon/2]} \bar{\delta} \left( \frac{1}{\hat{s}_m} \frac{\hat{\mu}(\sigma^{d_0}(C(m, z)))}{\hat{\mu}(\sigma^{d_0}(J_0))} \right)^t \\ &\leq c \bar{\delta} e^{-m[t\bar{s}+\varepsilon/2]} \left( \frac{1}{\hat{s}_m} \frac{\hat{\mu}(\sigma^{d_0}(C(m, z)))}{\hat{\mu}(\sigma^{d_0}(J_0))} \right)^t \end{aligned}$$

Hence, since  $\gamma_1 = \tilde{d}_1(\bar{s} + \varepsilon/(2t))$  we obtain (54) for  $j = 1$ .

Now, let us assume that we have already constructed the families  $\tilde{\mathcal{J}}_j$ ,  $\mathcal{J}_j$  and the numbers  $N_j$ ,  $\tilde{d}_j$ ,  $p(\tilde{d}_j)$  and  $d_j$  (for  $j = 1, \dots, m$ ), with  $p(\tilde{d}_j) \in \mathcal{I}(w)$ , related by

$$d_0 = \tilde{d}_0, \quad \tilde{d}_j = d_{j-1} + N_j \quad d_j = \tilde{d}_j + p(\tilde{d}_j)$$

in such way that

$$\sigma^{\tilde{d}_j}(\tilde{J}_j) = C(0, w), \quad \sigma^{\tilde{d}_j}(J_j) = C(p(\tilde{d}_j), w) \subset C(\ell_{\tilde{d}_j}, w), \quad \sigma^{d_j}(J_j) = C(0, \sigma^{p(\tilde{d}_j)}(w)), \quad (58)$$

and the properties (i), (ii) and (iii) of the theorem holds.

To construct  $\tilde{\mathcal{J}}_{m+1}$  we choose  $N_{m+1}$  such that:

- (i) proposition 3.1 holds for the measure  $\hat{\mathbf{m}}$  with  $N = M = N_{m+1}$ ,  $P_1 = C(0, \sigma^{p(\tilde{d}_m)}(w))$  and  $P_2 = C(0, w)$ ,
- (ii) There exists  $p(\tilde{d}_{m+1}) \in \mathcal{I}(w)$  with  $\tilde{d}_{m+1} := d_m + N_{m+1}$  such that  $\ell_{\tilde{d}_{m+1}} \leq p(\tilde{d}_{m+1}) < \ell_{\tilde{d}_{m+1}+1}$ ,

We define  $\tilde{\mathcal{J}}_{m+1}$  as

$$\tilde{\mathcal{J}}_{m+1} = \bigcup_{J \in \mathcal{J}_m} G_J^{-1}(\mathcal{S}_{N_{m+1}}(N_{m+1}, \varepsilon)).$$

and the family  $\mathcal{J}_{m+1}$  as

$$\mathcal{J}_{m+1} = \bigcup_{J \in \mathcal{J}_m} G_J^{-1}(\mathcal{L}_{m+1}) \quad \text{where} \quad \mathcal{L}_{m+1} = \bigcup_{S \in \mathcal{S}_{N_{m+1}}(N_{m+1}, \varepsilon)} G_S^{-1}(C(p(\tilde{d}_{m+1}), w))$$

In a similar way that in the initial step of the induction, one can check that by taking  $N_{m+1}$  large enough the properties (i),(ii) and (iii) of the theorem hold for  $j = m + 1$ . Notice that we need  $p(\tilde{d}_j) \in \mathcal{I}(w)$  in order to get (57) for  $j + 1$ , and (51) to get (iii). The pattern set  $\mathcal{Z}$  is contained in  $W_\sigma(\hat{P}, \ell_n, w)$  for (58).  $\square$

### 5.1.2 Lower bound without using other weak Gibbs measures

In this section we assume that  $\hat{\mu}$  is a *mixing* local weak  $\sigma$ -Gibbs measure with potential  $\phi$  such that  $\sum_{n \geq 1} V_n(\phi) < \infty$  and  $\phi \in L^1(\hat{\mu})$ . In particular, we have  $0 \leq P_G(\phi) - \int \phi d\hat{\mu} < \infty$  (see (46)). If  $\hat{\mu}$  is really weak (i.e  $\sup_n K_n = \infty$ ) then we ask the sequence  $\{\ell_n\}$  verifies  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ , this condition is not necessary if  $\sup_n K_n < \infty$ . Moreover, in the case of infinite countable alphabet and  $\hat{\mu}$  weak we will add the extra hypothesis  $\sup \phi < \infty$  in order to have for all  $z$  with  $z_0 = i_0$

$$\hat{\mu}(C(n, z)) \leq c(i_0) K_n \exp[-nP_G(\phi) + n \sup \phi],$$

and therefore for all  $\varepsilon > 0$  there exists  $n_0$  such that for all  $n \geq n_0$

$$\frac{1}{\hat{\mu}(C(n, z))} \geq \frac{1}{c(i_0)} \exp[n(P_G(\phi) - \sup \phi - \varepsilon)] \quad \text{for all } z \text{ with } z_0 = i_0 \quad (59)$$

By using the mixing properties of  $\hat{\mu}$  we get the following:

**Theorem 5.2.** *Let  $w$  be a  $\hat{\mu}$ -hitting point with sequence  $\mathcal{I}(w) = \{p_i\}$  and let*

$$\bar{s} := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\hat{\mu}(C(\ell_n, w))} < \infty. \quad (60)$$

*Then, for all  $\varepsilon > 0$  with*

$$P_G(\phi) - \int \phi d\hat{\mu} > \varepsilon$$

*there exists a  $(\tilde{\mathcal{J}}_j, J_j)$  pattern set  $\mathcal{Z}_\varepsilon$ , with  $\tilde{\mathcal{J}}_j$  a set of  $\tilde{d}_j$ -cylinders and  $J_j$  a set of  $d_j$ -cylinders, such that*

$$\mathcal{Z}_\varepsilon \subset \{z \in J_0 : \sigma^{\tilde{d}_j}(z) \in C(\ell_{\tilde{d}_j}, w) \text{ for } j \in \mathbb{N} \setminus \{0\}\} \subset W_\sigma(\hat{P}, \ell_n, w),$$

*and verifying:*

(i) *There exists  $\bar{\delta} > 0$  such that*

$$\hat{\mu}(\sigma^{d_j-1}(\tilde{\mathcal{J}}_j \cap J_{j-1})) := \sum_{\tilde{J}_j \in \tilde{\mathcal{J}}_j, \tilde{J}_j \subset J_{j-1}} \hat{\mu}(\sigma^{d_j-1}(\tilde{J}_j)) \geq \bar{\delta} \hat{\mu}(\sigma^{d_j-1}(J_{j-1})),$$

(ii)

$$e^{-(\tilde{d}_j - d_{j-1})(P_G(\phi) - \int \phi d\hat{\mu} + 2\varepsilon)} \leq \frac{\hat{\mu}(\sigma^{d_j-1}(\tilde{J}_j))}{\hat{\mu}(\sigma^{d_j-1}(J_{j-1}))} \leq e^{-(\tilde{d}_j - d_{j-1})(P_G(\phi) - \int \phi d\hat{\mu} - 2\varepsilon)}$$

(iii)

$$e^{-\varepsilon \tilde{d}_j} \hat{\mu}(C(d_j - \tilde{d}_j, w)) \leq \frac{\hat{\mu}(\sigma^{d_j-1}(J_j))}{\hat{\mu}(\sigma^{d_j-1}(\tilde{J}_j))} \leq e^{\varepsilon \tilde{d}_j} \hat{\mu}(C(d_j - \tilde{d}_j, w))$$

and

$$\widehat{\mu}(C(d_j - \widetilde{d}_j, w)) \geq e^{-\widetilde{d}_j(\overline{s} + \varepsilon)}$$

(iv) For some  $c > 0$

$$\widehat{s}_m \leq \frac{c}{\widehat{\mu}(C(m, z))^\varepsilon} \quad \text{for all } m \geq d_0 \quad \text{and } z \in \mathcal{Z}$$

with  $\{\widehat{s}_m\}$  the sequence in lemma 5.1

(v)

$$\lim_{j \rightarrow \infty} \frac{d_1 + \cdots + d_{j-1}}{\widetilde{d}_j} = 0$$

**Corollary 5.2.**

$$\text{Dim}_{\widehat{\mu}}(\mathcal{Z}_\varepsilon) \geq \frac{P_G(\phi) - \int \phi d\widehat{\mu} - 2\varepsilon}{P_G(\phi) - \int \phi d\widehat{\mu} + \overline{s} + 4\varepsilon}$$

*Proof.* Follows from last theorem and corollary 4.1 with  $\delta_j = \overline{\delta}$ ,  $\alpha_j = (\widetilde{d}_j - d_{j-1})(P_G(\phi) - \int \phi d\widehat{\mu} + 2\varepsilon)$ ,  $\beta_j = (\widetilde{d}_j - d_{j-1})(P_G(\phi) - \int \phi d\widehat{\mu} - 2\varepsilon)$  and  $\gamma_j = \widetilde{d}_j(\overline{s} + \varepsilon) + \varepsilon \widetilde{d}_j$ . Notice that, if  $\widehat{P} \subset C_i$ , then

$$\lim_{j \rightarrow \infty} \frac{(\alpha_1 + \cdots + \alpha_j) + (\gamma_1 + \cdots + \gamma_j)}{\widetilde{d}_j} = P_G(\phi) - \int \phi d\widehat{\mu} + \overline{s} + 4\varepsilon$$

and

$$\lim_{j \rightarrow \infty} \frac{\beta_1 + \cdots + \beta_j - (j+1) \log 1/\delta}{\widetilde{d}_j} = P_G(\phi) - \int \phi d\widehat{\mu} - 2\varepsilon.$$

□

By letting  $\varepsilon \rightarrow 0$  in the above corollary we obtain

**Corollary 5.3.**

$$\text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \geq \frac{P_G(\phi) - \int \phi d\widehat{\mu}}{P_G(\phi) - \int \phi d\widehat{\mu} + \overline{s}}.$$

**Remark 5.5.** If the variational principle holds then we have that  $P_G(\phi) - \int \phi d\widehat{\mu} \geq h_{\widehat{\mu}}$  and therefore

$$\text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \geq \frac{P_G(\phi) - \int \phi d\widehat{\mu}}{P_G(\phi) - \int \phi d\widehat{\mu} + \overline{s}} \geq \frac{h_{\widehat{\mu}}}{h_{\widehat{\mu}} + \overline{s}}.$$

and if  $\widehat{\mu}$  is an equilibrium measure  $P_G(\phi) - \int \phi d\widehat{\mu} = h_{\widehat{\mu}}$ .

We will use in section 6 the following remark to get Hausdorff dimension results for Markov transformations with infinite countable alphabet. His proof is given in section 5.1.3.

**Remark 5.6.** Under the hypothesis of theorem 5.2, if  $P_G(\phi) - \int \phi d\widehat{\mu} > 0$  and the collection  $\mathcal{P} = \{C(0, \sigma^{p_j}(w)) : p_j \in \mathcal{I}(w)\}$  is  $\varepsilon$ -uniformly  $\widehat{\mu}$ -good for all  $\varepsilon > 0$  small enough, then there exists a  $(\widetilde{\mathcal{I}}_j, \mathcal{J}_j)$  pattern set  $\mathcal{Z}_\varepsilon \subset W_\sigma(\widehat{P}, \ell_n, w)$  verifying the conditions in theorem 5.2 and with the following extra property:

For all

$$\gamma \geq \frac{2\varepsilon}{P_G(\phi) - \int \phi d\widehat{\mu} - \varepsilon} > 0$$

there is a constant  $c > 0$  such that for all  $z \in \mathcal{Z}$

$$\frac{\widehat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \geq c \quad \text{for } d_j \leq n < \widetilde{d}_{j+1}$$

By construction (also in theorem 5.2) we have that

$$\sigma^{\tilde{d}_{j+1}}(C(n, z)) = C(n - \tilde{d}_{j+1}, w) \quad \text{for} \quad \tilde{d}_{j+1} \leq n \leq d_{j+1}$$

Recall that if  $\#\mathcal{P} < \infty$  then we have that  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good (since  $\hat{\mu}$  is mixing). For  $\#\mathcal{P}$  infinite, if  $\hat{\mu}$  has summable uniform rate of mixing of order 3 in  $\mathcal{P}$  and  $\phi \in L^2(\hat{\mu})$ , then  $\mathcal{P}$  is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good (see corollary 3.1).

*Proof of theorem 5.2.* We construct the  $(\tilde{\mathcal{J}}, \mathcal{J})$ -pattern set  $\mathcal{Z}_\varepsilon$  in a similar way as in theorem 5.1, but now we use the measure  $\hat{\mu}$  instead of the measure  $\hat{\mathbf{m}}$  when we apply proposition 3.1. Since the Markov chain is topologically mixing we have (as in the proof of theorem 5.1) that there exists a cylinder  $\tilde{J}_0 \subset \hat{P}$  such that  $\tilde{J}_0 \in \hat{\mathcal{P}}_{\tilde{d}_0}$  and  $\sigma^{\tilde{d}_0}(\tilde{J}_0) = C(0, w)$ . We define  $J_0 := \tilde{J}_0$  and  $\mathcal{J}_0 = \tilde{\mathcal{J}}_0 = \{\tilde{J}_0\}$ . Moreover, let  $i_0$  be the first symbol of the cylinder  $\hat{P}$ . In the case  $\sup K_n = \infty$  we are assuming that  $\sup \phi < \infty$ , and so we can choose  $d_0$  large enough so that for all  $m \geq d_0$

$$\hat{s}_m \leq e^{m\varepsilon[P_G(\phi) - \sup \phi - \varepsilon]/2} \leq \left( \frac{1}{c(i_0)} e^{m[P_G(\phi) - \sup \phi - \varepsilon]} \right)^\varepsilon \quad (61)$$

and so by (59) we know that the condition (iv) holds for all  $m \geq d_0$ . Of course, in the case  $\sup K_n < \infty$ , the condition (iv) holds for all  $z$  (with  $\varepsilon = 0$ ) and the hypothesis  $\sup \phi < \infty$  is not required.

To construct the family  $\tilde{\mathcal{J}}_1$  we apply proposition 3.1 for the measure  $\hat{\mu}$  with  $P_1 = P_2 = C(0, w)$ . Let  $N_1$  be a natural number large enough so that  $\mathcal{S}_N(M, \varepsilon)$  with  $N = M = N_1$  satisfies (20). We define  $\tilde{\mathcal{J}}_1$  as the family of cylinders  $G_{J_0}^{-1}(S)$  with  $S \in \mathcal{S}_{N_1}(N_1, \varepsilon)$ . Notice that by proposition 3.1

$$\hat{\mu}(\sigma^{d_0}(\tilde{\mathcal{J}}_1 \cap J_0)) := \sum_{\tilde{J}_1 \in \tilde{\mathcal{J}}_1} \hat{\mu}(\sigma^{d_0}(\tilde{J}_1)) \geq \bar{\delta} \hat{\mu}(\sigma^{d_0}(J_0)), \quad \text{with} \quad \bar{\delta} = \frac{1}{2} \hat{\mu}(C(0, w)).$$

and, by taking  $N_1$  large enough, from property (18) we have for all  $\tilde{J}_1 \in \tilde{\mathcal{J}}_1$  that

$$e^{-(\tilde{d}_1 - d_0)(P_G(\phi) - \int \phi d\hat{\mu} + 2\varepsilon)} \leq \frac{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))}{\hat{\mu}(\sigma^{d_0}(J_0))} \leq e^{-(\tilde{d}_1 - d_0)(P_G(\phi) - \int \phi d\hat{\mu} - 2\varepsilon)} \quad \text{with} \quad \tilde{d}_1 = d_0 + N_1.$$

In order to get the condition (v) we will also ask  $N_1$  be large enough so that  $\tilde{d}_1 \geq 2d_0$ .

Let  $\mathcal{I}(w)$  denote the sequence given by the definition of  $\hat{\mu}$ -hitting point. We can also choose  $N_1$  so that there exists  $p_i \in \mathcal{I}(w)$  such that  $\ell_{\tilde{d}_1} \leq p_i < \ell_{\tilde{d}_1+1}$ . We denote this natural number  $p_i$  by  $p(\tilde{d}_1)$ .

In each set  $S$  in  $\mathcal{S}_{N_1}(N_1, \varepsilon)$  we take the subset  $L := G_S^{-1}(C(p(\tilde{d}_1), w))$  and we denote by  $\mathcal{L}_1$  this family of sets. We define the family  $\mathcal{J}_1$  as the collection  $G_{J_0}^{-1}(L)$  with  $L \in \mathcal{L}_1$ . By construction for all  $J_1 \in \mathcal{J}_1$  there exists an unique, then  $\tilde{J}_1 \in \tilde{\mathcal{J}}_1$  such that  $J_1 \subset \tilde{J}_1$ ,

$$\sigma^{\tilde{d}_1}(\tilde{J}_1) = C(0, w), \quad \sigma^{\tilde{d}_1}(J_1) = C(p(\tilde{d}_1), w) \subset C(t_{\tilde{d}_1}, w), \quad \text{and} \quad \sigma^{d_1}(J_1) = C(0, \sigma^{p(\tilde{d}_1)}(w)),$$

with  $\tilde{d}_1 = d_0 + N_1$  and  $d_1 := \tilde{d}_1 + p(\tilde{d}_1)$ . Also, we have from lemma 5.1, (we may assume that  $0 \in \mathcal{I}(w)$ ), that

$$\frac{1}{\hat{s}_{N_1+p(\tilde{d}_1)}} \frac{\hat{\mu}(C(p(\tilde{d}_1), w))}{\hat{\mu}(C(0, w))} \leq \frac{\hat{\mu}(\sigma^{d_0}(J_1))}{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))} \leq \hat{s}_{N_1+p(\tilde{d}_1)} \frac{\hat{\mu}(C(p(\tilde{d}_1), w))}{\hat{\mu}(C(0, w))} \quad (62)$$

Recall that  $p(\tilde{d}_1) < \ell_{\tilde{d}_1+1}$ . So, from (60) and by using that  $\lim_{m \rightarrow \infty} \frac{1}{m} \log \hat{s}_m = 0$ , and  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$  if  $\sup_n K_n = \infty$ , we have for  $N_1$  large enough that

$$e^{-\varepsilon \tilde{d}_1} \hat{\mu}(C(p(\tilde{d}_1), w)) \leq \frac{\hat{\mu}(\sigma^{d_0}(J_1))}{\hat{\mu}(\sigma^{d_0}(\tilde{J}_1))} \leq e^{\varepsilon \tilde{d}_1} \hat{\mu}(C(p(\tilde{d}_1), w))$$



and

$$\widehat{\mu}(C(p(\widetilde{d}_1), w)) \geq \widehat{\mu}(C(\ell_{\widetilde{d}_1+1}, w)) \geq e^{-\widetilde{d}_1(\overline{s}+\varepsilon)}.$$

The inductive step follows as in the proof of theorem 5.1; just remember that we use proposition 3.1 for the measure  $\widehat{\mu}$  (instead of  $\widehat{\mathbf{m}}$ ). Notice that property (iv) holds due to (61) and property (v) follows because we choose  $N_{m+1}$  large enough so that  $\widetilde{d}_{m+1} = d_m + N_{m+1} \geq 2(m+1)(d_0 + \dots + d_m)$ .  $\square$

### 5.1.3 Proofs of Remarks 5.4 and 5.6

*Proof of Remark 5.4.* The construction of the pattern set  $\mathcal{Z}_\varepsilon$  is similar to the one realized in the proof of theorem 5.1. The difference is that, by corollary 3.1, we can substitute the families  $S_{N_j}(N_j, \varepsilon)$  by the families  $S_{N_j}(m_0, \varepsilon)$  with  $m_0$  is a fixed natural number.

For  $d_j \leq n < \widetilde{d}_{j+1}$  with  $z \in \mathcal{Z}_\varepsilon$ , we have that

$$S \subset \sigma^{d_j}(C(n, z)) \subset P_1 := C(0, \sigma^{p(\widetilde{d}_j)}(w)) \text{ for some } S \in S_{N_{j+1}}(m_0, \varepsilon)$$

and so  $\sigma^{d_j}(C(n, z)) = C(n - d_j, u) \subset P_1$  for some  $u \in \text{Good}(m_0, \varepsilon)$ .

Therefore, for  $n - d_j \geq m_0$

$$\frac{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \geq \frac{e^{-(n-d_j+1)(P_G(t\phi)-t \int \phi d\widehat{\mathbf{m}}+\varepsilon)}}{e^{-(n-d_j)(P_G(t\phi)-t \int \phi d\widehat{\mathbf{m}}-\varepsilon)(1+\gamma)}}, \quad (63)$$

and for  $0 \leq n - d_j < m_0$

$$\frac{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \geq \frac{\widehat{\mathbf{m}}(C(m_0, u))}{\widehat{\mathbf{m}}(P_1)^{1+\gamma}} \geq \widehat{\mathbf{m}}(C(m_0, u)) \geq e^{-m_0(P_G(t\phi)-t \int \phi d\widehat{\mathbf{m}}+\varepsilon)}. \quad (64)$$

From lemma 5.3 and by using that  $w$  is a  $\widehat{\mu}$ -hitting point (recall that  $\sigma^{d_j}(C(n, z)) \subset P_1 = C(0, \sigma^{p(\widetilde{d}_j)}(w))$ ), the condition (51) holds and  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \widehat{k}_n = 0$ , we get the following estimation with  $c$  a positive constant

$$\left( \frac{\widehat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \right)^t \geq c e^{-(n-d_j)[\gamma(P_G(t\phi)-tP_G(\phi))+\varepsilon]} \frac{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mathbf{m}}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \quad (65)$$

Let  $v := \sigma^{\widetilde{d}_j}(z)$ . Since  $\sigma^{p(\widetilde{d}_j)}(v) = \sigma^{d_j}(z) \in C(0, \sigma^{p(\widetilde{d}_j)}(w))$ , from lemma 5.1 we have that

$$\begin{aligned} \widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n+1, z))) &= \widehat{\mu}(C(n+1 - \widetilde{d}_j, v)) \geq \frac{1}{\widehat{s}_{n+1-\widetilde{d}_j}} \frac{\widehat{\mu}(\sigma^{p(\widetilde{d}_j)}(C(n+1 - \widetilde{d}_j, v)))}{\widehat{\mu}(\sigma^{p(\widetilde{d}_j)}(C(p(\widetilde{d}_j), v)))} \widehat{\mu}(C(p(\widetilde{d}_j), v)) \\ &= \frac{1}{\widehat{s}_{n+1-\widetilde{d}_j}} \frac{\widehat{\mu}(C(p(\widetilde{d}_j), v))}{\widehat{\mu}(\sigma^{p(\widetilde{d}_j)}(C(p(\widetilde{d}_j), v)))} \widehat{\mu}(\sigma^{d_j}(C(n+1, z))) \end{aligned}$$

and also

$$\widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n, z))) = \widehat{\mu}(C(n - \widetilde{d}_j, v)) \leq \widehat{s}_{n-\widetilde{d}_j} \frac{\widehat{\mu}(C(p(\widetilde{d}_j), v))}{\widehat{\mu}(\sigma^{p(\widetilde{d}_j)}(C(p(\widetilde{d}_j), v)))} \widehat{\mu}(\sigma^{d_j}(C(n, z)))$$

Hence,

$$\left( \frac{\widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq \frac{1}{(\widehat{s}_{n+1-\widetilde{d}_j} \widehat{s}_{n-\widetilde{d}_j}^{1+\gamma})^t} \left( \frac{\widehat{\mu}(\sigma^{p(\widetilde{d}_j)}(C(p(\widetilde{d}_j), v)))}{\widehat{\mu}(C(p(\widetilde{d}_j), v))} \right)^{\gamma t} \left( \frac{\widehat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \right)^t$$

We use again lemma 5.1, and also that  $C(p(\widetilde{d}_j) - 1, v) = C(p(\widetilde{d}_j) - 1, w)$ , since  $v \in C(p(\widetilde{d}_j), w)$ , and we obtain

$$\left( \frac{\widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{\widetilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq \frac{1}{[\widehat{s}_{n+1-\widetilde{d}_j} \widehat{s}_{n-\widetilde{d}_j}^{1+\gamma} \widehat{s}_{d_j-\widetilde{d}_j}^\gamma]^t} \frac{1}{(\widehat{\mu}(C(p(\widetilde{d}_j) - 1, w)))^{\gamma t}} \left( \frac{\widehat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\widehat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \right)^t$$

Since  $p(\tilde{d}_j) \geq \ell_{\tilde{d}_j}$ ,  $\liminf_{n \rightarrow \infty} \hat{\mu}(C(n+1, w)) / \hat{\mu}(C(n, w)) > 0$  and  $\underline{s} = \liminf_{n \rightarrow \infty} -(\log \hat{\mu}(C(\ell_n, w))) / n$  we have

$$\hat{\mu}(C(p(\tilde{d}_j) - 1, w)) \leq \hat{\mu}(C(\ell_{\tilde{d}_j} - 1, w)) \leq c_1 e^{-\tilde{d}_j(\underline{s} - \varepsilon)}$$

for some positive constant  $c_1$ . And as  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \hat{s}_n = 0$  we get that

$$\left( \frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq c_2 e^{-(n-\tilde{d}_j)\varepsilon(1+\gamma)t} e^{\tilde{d}_j(\underline{s} - \varepsilon)\gamma t} \left( \frac{\hat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \right)^t$$

for some  $c_2 > 0$ . We recall that  $p(\tilde{d}_j) \leq \ell_{\tilde{d}_j+1}$  and  $\limsup_{n \rightarrow \infty} \ell_n / n < \infty$ , and so

$$n - \tilde{d}_j = n - d_j + p(\tilde{d}_j) < n - d_j + \ell_{\tilde{d}_j+1} \leq n - d_j + c \tilde{d}_j$$

for some positive constant  $c$ . Therefore since  $\underline{s} > 0$  for  $\varepsilon > 0$  small enough we have that

$$\left( \frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq c_2 e^{-(n-d_j)\varepsilon(1+\gamma)t} \left( \frac{\hat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \right)^t$$

By using (63), (64) and (65) in the above inequality we get the following:

For  $n - d_j \geq m_0$

$$\left( \frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq c e^{(n-d_j)[\gamma t(P_G(\phi) - \int \phi d\hat{\mathbf{m}}) - (3+\gamma-(1+\gamma)t)\varepsilon]} \geq c,$$

and for  $0 \leq n - d_j < m_0$

$$\left( \frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \right)^t \geq c e^{-m_0[(\gamma+1)P_G(t\phi) - \gamma t P_G(\phi) - t \int \phi d\hat{\mathbf{m}} + 2\varepsilon + \varepsilon(1+\gamma)t]} = c t t e$$

Finally, notice that for  $\tilde{d}_{j+1} \leq n < d_{j+1}$  with  $z \in \mathcal{Z}_\varepsilon$ , we have that

$$C(p(\tilde{d}_{j+1}), w) \subset \sigma^{\tilde{d}_{j+1}}(C(n, z)) \subset C(0, w)$$

and so  $\sigma^{\tilde{d}_{j+1}}(C(n, z)) = C(n - \tilde{d}_{j+1}, w)$  and  $\sigma^{\tilde{d}_{j+1}}(C(n+1, z)) = C(n+1 - \tilde{d}_{j+1}, w)$ . □

*Proof of Remark 5.6.* The construction of the pattern set  $\mathcal{Z}_\varepsilon$  is similar to the one realized in the proof of theorem 5.2. The difference is that, by corollary 3.1, we can substitute the families  $S_{N_j}(N_j, \varepsilon)$  by the families  $S_{N_j}(m_0, \varepsilon)$  with  $m_0$  is a fixed natural number.

For  $d_j \leq n < \tilde{d}_{j+1}$  with  $z \in \mathcal{Z}_\varepsilon$  we have that

$$S \subset \sigma^{d_j}(C(n, z)) \subset P_1 := C(0, \sigma^{p(\tilde{d}_j)}(w)) \text{ for some } S \in S_{N_{j+1}}(m_0, \varepsilon)$$

and so  $\sigma^{d_j}(C(n, z)) = C(n - d_j, u) \subset P_1$  for some  $u \in \text{Good}(m_0, \varepsilon)$ .

Therefore, for  $n - d_j \geq m_0$

$$\frac{\hat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \geq \frac{1}{c(w_{p(\tilde{d}_j)})^{2+\gamma}} \frac{e^{-(n+1-d_j)(P_G(\phi) - \int \phi d\hat{\mu} + \varepsilon)}}{e^{-(n-d_j)(P_G(\phi) - \int \phi d\hat{\mu} - \varepsilon)(1+\gamma)}} \geq c e^{(n-d_j)[\gamma(P_G(\phi) - \int \phi d\hat{\mu}) - \varepsilon(2+\gamma)]} \geq c$$

and for  $0 \leq n - d_j < m_0$

$$\frac{\hat{\mu}(\sigma^{d_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{d_j}(C(n, z)))^{1+\gamma}} \geq \frac{\hat{\mu}(C(m_0, u))}{\hat{\mu}(P_1)^{1+\gamma}} \geq e^{-m_0(P_G(\phi) - \int \phi d\hat{\mu} + \varepsilon)}$$

Finally, notice that for  $\tilde{d}_{j+1} \leq n < d_{j+1}$  with  $z \in \mathcal{Z}_\varepsilon$  we have that

$$C(p(\tilde{d}_{j+1}), w) \subset \sigma^{\tilde{d}_{j+1}}(C(n, z)) \subset C(0, w)$$

and so  $\sigma^{\tilde{d}_{j+1}}(C(n, z)) = C(n - \tilde{d}_{j+1}, w)$  and  $\sigma^{\tilde{d}_{j+1}}(C(n+1, z)) = C(n+1 - \tilde{d}_{j+1}, w)$ . □

## 5.2 Special cases

Along this section we assume that

$$s := \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\hat{\mu}(C(\ell_n, w))} < \infty.$$

If  $\hat{\mu}$  is weak (i.e  $\sup_n K_n = \infty$ ), then we also require  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ .

We recall (see remark 5.2) that if  $s = \infty$ , then the target-ball set has zero  $\hat{\mu}$ -dimension. Also if  $\lim_{n \rightarrow \infty} \ell_n/n = \infty$  and  $\hat{\mu}$  is ergodic,  $\phi \in L^1(\hat{\mu})$  and  $\sum_{n \geq 1} V_n(\phi) < \infty$ , then by (46) we have that  $s = \infty$  for  $w$   $\hat{\mu}$ -a.e.

### 5.2.1 Markov chain with finite alphabet

For  $\phi$  continuous (with  $P_{top}(\phi) < \infty$ ) the function  $G(t) := P_{top}(t\phi) - P_{top}(\phi)t$  is continuous, convex, decreasing and  $G(1) = 0$ . Therefore for the results in the previous section we have for all  $w \in \Sigma_A^{\mathcal{I}}$  the following:

**Theorem 5.3.** *If the alphabet is finite and for all  $0 < t \leq 1$  there is a mixing weak  $\sigma$ -Gibbs measure with continuous potential  $t\phi$ , then*

$$\text{Dim}_{\hat{\mu}}(W_{\sigma}(\hat{P}, \ell_n, w)) = T$$

with  $T$  the root of the equation

$$P_{top}(t\phi) - P_{top}(\phi)t = st$$

If  $\hat{\mu}$  is mixing, then

$$T \geq \frac{P_{top}(\phi) - \int \phi d\hat{\mu}}{P_{top}(\phi) - \int \phi d\hat{\mu} + s} \geq \frac{h_{\hat{\mu}}}{h_{\hat{\mu}} + s}.$$

and moreover, if  $\hat{\mu}$  is the (unique) equilibrium measure of the potential  $\phi$  (which is exact and Gibbs), then  $P_{top}(\phi) - \int \phi d\hat{\mu} = h_{\hat{\mu}}$ . By Shannon-MacMillan-Breiman theorem if  $v = \lim_{n \rightarrow \infty} \ell_n/n$  then  $s = vh_{\hat{\mu}}$  for  $w$   $\hat{\mu}$ -a.e.

**Remark 5.7.** From classical results of Bowen and Ruelle (see [8], [40]) one knows that for finite alphabet there is a unique equilibrium measure for any potential with the Walters property. Moreover, this measure is a  $\sigma$ -Gibbs measure and it is exact (whence ergodic and strong mixing). So the hypothesis of above theorem holds for these potentials.

We refers to section 5.2.3 where we consider non-Hölder potentials  $\phi$  such that there exists a mixing weak  $\sigma$ -Gibbs measure with potential  $t\phi$ .

### 5.2.2 Markov chain with infinite alphabet but BI property

**Proposition 5.2.** *Let  $\Sigma_A^{\mathcal{I}}$  be a topologically mixing Markov chain with infinite countable alphabet satisfying the BI property and  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  be a positive recurrent potential with  $P_G(\phi) < \infty$  and such that  $V_1(\phi) < \infty$  and  $\phi$  is weak Hölder continuous. Then the Ruelle-Perron-Frobenius measure  $\hat{\mathbf{m}}$  verifies*

$$0 < \frac{1}{c} \inf \{h(x) : x \in C(0, z)\} \leq \frac{\hat{\mathbf{m}}(C(n-1, z))}{\exp(-nP_G(\phi) + \sum_{j=0}^{n-1} \phi \circ \sigma^j(z))} \leq c$$

with  $c > 1$  depending on the finite collection of symbols given by the BI property and  $h$  a positive continuous function such that  $\log h$  and  $\log h \circ \sigma$  are weak Hölder continuous and  $V_1(\log h) < \infty$ . In particular  $\hat{\mathbf{m}}$  is local  $\sigma$ -Gibbs.

*Proof.* By Sarig results (see [45] or [43]) we have that  $d\hat{\mathbf{m}} = h d\nu$  with  $h$  a positive continuous function and  $\nu$  a conservative measure such that  $L_{\phi}h = \lambda h$  and  $L_{\phi}^*\nu = \lambda\nu$  with  $\lambda = \exp P_G(\phi)$  and  $L_{\phi}$  de Ruelle operator. The measure  $\nu$  is finite and positive on cylinders and  $\log h$  and  $\log h \circ \sigma$  are weak Hölder continuous and  $V_1(\log h) < \infty$ .

Proceeding as in [43] (or in [47] p. 98) by using that  $h$  is bounded away from zero and infinity on 0-cylinders and  $\sum_{n=1}^{\infty} V_n(\phi) < \infty$  we have that

$$\widehat{\mathbf{m}}(C(n-1, z)) \asymp \frac{1}{\lambda^n} \int L_\phi^n \mathbb{1}_{C(n-1, z)} d\nu \asymp \frac{1}{\lambda^n} \exp\left(\sum_{j=0}^{n-1} \phi \circ \sigma^j(z)\right) \nu(\sigma^n(C(n-1, z)))$$

with comparability constants depending on the bounds of  $h$  in  $C(0, z)$ . And since we have the BI property, then

$$\nu(\sigma^n(C(n-1, z))) = \nu(\sigma(C(0, \sigma^{n-1}(z)))) \geq \inf\{\nu(C_k) : k \in \mathcal{I}_0\} > 0$$

where  $\mathcal{I}_0$  is the finite collection of symbols given by the BI property. (Sarig also proved that the BI property implies that  $\sup h < \infty$ )

To conclude that  $\widehat{\mathbf{m}}$  is local Gibbs we will see that  $\nu(\sigma(C(0, \sigma^{n-1}(z))))$  is upper bounded by a positive constant (depending on  $\mathcal{I}_0$  and  $z_0$ ). By the BI property we know that there exists  $k \in \mathcal{I}_0$  such that  $C_{z_{n-1}k} \neq \emptyset$ , also since the Markov chain is topologically mixing given  $k \in \mathcal{I}_0$  and  $z_0$  there exists two finite sequences  $u \in \Sigma_A^{\mathcal{I}}|_m$ ,  $v \in \Sigma_A^{\mathcal{I}}|_s$  such that  $u_0 = v_0 = v_s = k$ ,  $u_m = z_0$ , and  $u_i, v_j \neq k$  for  $1 \leq i \leq m-1$ ,  $1 \leq j \leq s-1$ .

For all  $w \in \Sigma_A^{\mathcal{I}}$  and  $\ell > m+n$  we have that

$$\lambda^{n+m} \nu(C(\ell+s-1, w)) = \int L_\phi^{n+m} \mathbb{1}_{C(\ell+s-1, w)} d\nu \asymp \exp\left(\sum_{j=0}^{m+n-1} \phi \circ \sigma^j(w)\right) \nu(\sigma^{m+n}(C(\ell+s-1, w)))$$

and also

$$\lambda^{n+m} \nu(C(m+n-1, w)) = \int L_\phi^{n+m} \mathbb{1}_{C(m+n-1, w)} d\nu \asymp \exp\left(\sum_{j=0}^{m+n-1} \phi \circ \sigma^j(w)\right) \nu(\sigma^{m+n}(C(m+n-1, w)))$$

with absolute constants of comparability. Hence, if  $w \in \Sigma_A^{\mathcal{I}}$  has the property

$$w|_{m+n} = (k, u_1, \dots, u_{m-1}, z_0, \dots, z_{n-1}, k) \quad (66)$$

then  $\nu(\sigma(C(0, \sigma^{n-1}(z)))) = \nu(\sigma^{m+n}(C(m+n-1, w)))$  and we have

$$\nu(\sigma(C(0, \sigma^{n-1}(z)))) \asymp \frac{\nu(C(\ell+s-n-m-1, \sigma^{m+n}(w))) \nu(C(m+n-1, w))}{\nu(C(\ell+s-1, w))}.$$

Therefore

$$\nu(\sigma(C(0, \sigma^{n-1}(z)))) \asymp \frac{\widehat{\mathbf{m}}(C(\ell+s-n-m-1, \sigma^{m+n}(w))) \widehat{\mathbf{m}}(C(m+n-1, w))}{\widehat{\mathbf{m}}(C(\ell+s-1, w))} \quad (67)$$

with comparability constants depending on the bounds of  $h$  in the finite collection of the 0-cylinders  $C_k$  with  $k \in \mathcal{I}_0$ .

Now fix  $p, \ell \in \mathbb{N}$  with  $\ell > m+n+p$  and consider the points  $w \in \Sigma_A^{\mathcal{I}}$  verifying (66) and with the properties:

- (a)  $\sigma^\ell(w)|_{s-1} = v|_{s-1} = (k, v_1, \dots, v_{s-1})$
- (b)  $\#\{w_i : w_i = k \text{ with } m+n < i < \ell\} = p$

Note that we can write (67) as

$$\nu(\sigma(C(0, \sigma^{n-1}(z)))) \widehat{\mathbf{m}}(C(\ell+s-1, w)) \asymp \widehat{\mathbf{m}}(\sigma^{n+m}(C(\ell+s-1, w))) \widehat{\mathbf{m}}(C(m+n-1, w))$$

and for all  $w$  verifying (66)  $C(m+n-1, w)$  is the cylinder  $A := C_{ku_1 \dots u_{m-1} z_0 \dots z_{n-1}}$ . By adding all over  $w \in \Sigma_A^{\mathcal{I}}$  with  $\sigma^{\ell+s}(w) = w$  and verifying (66) and properties (a) and (b) we get that

$$\nu(\sigma(C(0, \sigma^{n-1}(z))) \asymp \frac{\widehat{\mathbf{m}}[B_\ell \cap \sigma^{-(\ell-(m+n))}(C(s-1, v))]}{\widehat{\mathbf{m}}[A \cap \sigma^{-(m+n)}(B_\ell) \cap \sigma^{-\ell}(C(s-1, v))]} \widehat{\mathbf{m}}(A)$$

with  $B_\ell := \{(y_0, y_1, \dots) \in \Sigma_A^{\mathcal{I}} : y_0 = k \text{ and } \#\{y_i : y_i = k \text{ with } 0 < i < \ell - m - n\} = p\}$ . Hence for  $p$  fixed

$$\nu(\sigma(C(0, \sigma^{n-1}(z))) \asymp \frac{\widehat{\mathbf{m}}[\bigcup_{\ell > m+n} B_\ell \cap \sigma^{-(\ell-(m+n))}(C(s-1, v))]}{\widehat{\mathbf{m}}[A \cap \bigcup_{\ell > m+n} \sigma^{-(m+n)}(B_\ell) \cap \sigma^{-\ell}(C(s-1, v))]} \widehat{\mathbf{m}}(A) \quad (68)$$

Let  $\bar{\sigma} : C_k \rightarrow C_k$  denote the induced first return map on the 0-cylinder  $C_k$ , i.e.

$$\bar{\sigma}(x) = \sigma^{\varphi(x)}(x) \quad \text{with} \quad \varphi(x) = \mathbb{1}_{C_k} \inf\{n > 0 : \sigma^n(x) \in C_k\}$$

and let  $\mathcal{S} := \{w|_n : n \geq 0, w_0 = k, w_i \neq k \text{ for } 0 < i \leq n, (w_0, w_1, \dots, w_n, k) \in \Sigma_A^{\mathcal{I}}|_{n+1}\}$  the set of induced symbols. Then  $\bar{v} := v|_{s-1} \in \mathcal{S}$ , and  $A$  is a cylinder in the topological Markov chain  $\Sigma^{\mathcal{S}}$ . If we denote the cylinders in  $\Sigma^{\mathcal{S}}$  by  $\bar{C}(n, \cdot)$ , and  $A$  is the cylinder  $\bar{C}(r-1, x)$  (for some  $r \in \mathbb{N}$  and  $x \in \Sigma^{\mathcal{S}}$ ), then we can re-write (68) as

$$\nu(\sigma(C(0, \sigma^{n-1}(z))) \asymp \frac{\overline{\mathbf{m}}(\bar{\sigma}^{-(p+1)}(\bar{C}(0, \bar{v}))) \overline{\mathbf{m}}(\bar{C}(r-1, x))}{\overline{\mathbf{m}}[\bar{C}(r-1, x) \cap \bar{\sigma}^{-(r+p+1)}(\bar{C}(0, \bar{v}))]} = \frac{\overline{\mathbf{m}}(\bar{C}(0, \bar{v})) \overline{\mathbf{m}}(\bar{C}(r-1, x))}{\overline{\mathbf{m}}[\bar{C}(r-1, x) \cap \bar{\sigma}^{-(r+p+1)}(\bar{C}(0, \bar{v}))]}$$

where  $\overline{\mathbf{m}}$  is the  $\bar{\sigma}$ -invariant measure  $\widehat{\mathbf{m}}_k \circ \bar{\pi}$  with  $\widehat{\mathbf{m}}_k$  the normalized restriction of  $\widehat{\mathbf{m}}$  to  $C_k$  and  $\bar{\pi} : \Sigma^{\mathcal{S}} \rightarrow C_k \subset \Sigma_A^{\mathcal{I}}$  the natural injection. But in [45] Sarig obtained (by using results of Aaronson, Denker and Urbanski [3] on the Schweiger property) that  $\bar{\sigma}$  is exponentially continued fraction mixing (see remark 3.6), and therefore for  $p$  large

$$\overline{\mathbf{m}}[\bar{C}(r-1, x) \cap \bar{\sigma}^{-(r+p+1)}(\bar{C}(0, \bar{v}))] \geq (1 - c\theta^{p+1}) \overline{\mathbf{m}}(\bar{C}(0, \bar{v})) \overline{\mathbf{m}}(\bar{C}(r-1, x))$$

for some constant  $c > 1$  and  $\theta \in (0, 1)$ . So, we obtain that  $\nu(\sigma(C(0, \sigma^{n-1}(z)))$  is upper bounded by a constant depending on  $z_0$  (and on the finite family  $\mathcal{I}_0$ ).  $\square$

**Theorem 5.4.** *If  $\Sigma_A^{\mathcal{I}}$  satisfies the BI property,  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  is weak Hölder continuous such that  $V_1(\phi) < \infty$  and  $\phi$  is positive recurrent then, for  $w$   $\widehat{\mu}$ -a.e*

$$T^- \leq \text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \leq T^+$$

with

$$T^- = \sup\{t \in (0, 1] : \infty > P_G(t\phi) - P_G(\phi)t > st, t\phi \text{ positive recurrent and } \phi \in L^1(\widehat{\mathbf{m}}_t)\}$$

$$T^+ = \inf\{t \in (0, 1] : P_G(t\phi) - P_G(\phi)t < st\}$$

Here  $\widehat{\mathbf{m}}_t$  is the RPF measure for the potential  $t\phi$ .

**Remark 5.8.** *By results in section 5.1.2 we also know that if  $\widehat{\mu}$  is mixing with potential  $\phi$  such that  $\sum_{n=1}^\infty V_n(\phi) < \infty$ ,  $\phi \in L^1(\mu)$ , and  $\sup \phi < \infty$  in the case  $\widehat{\mu}$  is not Gibbs (i.e  $\sup_n K_n = \infty$ ), then*

$$\text{Dim}_{\widehat{\mu}}(W_\sigma(\widehat{P}, \ell_n, w)) \geq \frac{P_G(\phi) - \int \phi d\widehat{\mu}}{P_G(\phi) - \int \phi d\widehat{\mu} + s} \quad (69)$$

*In particular, if  $\widehat{\mu}$  is the RPF measure of a positive recurrent potential  $\phi$  with summable variations such that  $\sup \phi < \infty$  and  $P_G(\phi) < \infty$ , and  $h_{\widehat{\mu}} < \infty$ , then (see Sarig's results in section 3.1)  $\widehat{\mu}$  is an equilibrium measure which is exact (whence ergodic and strong mixing). Hence  $P_G(\phi) - \int \phi d\widehat{\mu} = h_{\widehat{\mu}}$ .*

*Proof.* By Sarig's results (see [45]) we know that there is a finite RPF measure ( $\widehat{\mathbf{m}}_t$ ) for the (positive recurrent) potential  $t\phi$  which is  $\sigma$ -invariant and exact and by lemma 5.2 it is a local  $\sigma$ -Gibbs measure. Also, we know that condition (51) holds in a set of full  $\widehat{\mu}$ -measure (see remark 5.1). This allow us to get the result from the theorems on section 5.1.  $\square$

If  $\Sigma_A^{\mathcal{I}}$  satisfies the stronger property BIP, then any potential  $\phi$  with the Walters condition and  $V_1(\phi) < \infty$  and  $P_G(\phi) < \infty$  is positive recurrent, and the RPF measure is a  $\sigma$ -Gibbs measure exact (see [46], [47]). Notice that since measures are not local, then (51) always holds. Hence in this case we get the following:

**Theorem 5.5.** *If  $\Sigma_A^{\mathcal{I}}$  satisfies the BIP property, the Walters condition holds for  $\phi$ ,  $V_1(\phi) < \infty$  and  $P_G(\phi) < \infty$ , then for all  $w$  (with  $s < \infty$ )*

$$T^- \leq \text{Dim}_{\hat{\mu}}(W_{\sigma}(\hat{P}, \ell_n, w)) \leq T^+$$

with

$$\begin{aligned} T^- &= \sup\{t \in (0, 1] : \infty > P_G(t\phi) - P_G(\phi)t > st \text{ and } \phi \in L^1(\hat{\mathbf{m}}_t)\} \\ T^+ &= \inf\{t \in (0, 1] : P_G(t\phi) - P_G(\phi)t < st\} \end{aligned}$$

Here  $\hat{\mathbf{m}}_t$  is the RPF measure for the potential  $t\phi$ .

**Remark 5.9.** *If*

$$-\sum_{i \in \mathcal{I}} \hat{\mathbf{m}}_t(C_i) \log \hat{\mathbf{m}}_t(C_i) < \infty \quad (70)$$

then  $\hat{\mathbf{m}}_t$  has finite entropy. But, if BIP holds and  $\phi$  has summable variations,  $\sup \phi < \infty$ ,  $P_G(t\phi) < \infty$ , and  $\hat{\mathbf{m}}_t$  has finite entropy, then  $\phi \in L^1(\hat{\mathbf{m}}_t)$  and  $\hat{\mathbf{m}}_t$  is an equilibrium measure for  $t\phi$  (see section 3 for references). Hence, under these hypothesis, if there exists  $0 < t_1 \leq 1$  such that  $\infty > P_G(t_1\phi) - P_G(\phi)t_1 > st_1$  and (70) holds for all  $t_1 \leq t \leq 1$ , then

$$T^- = \sup\{t \in [t_1, 1] : P_G(t\phi) - P_G(\phi)t > st\}$$

If  $\hat{\mu}$  is  $\sigma$ -Gibbs (for example if it is the RPF measure for  $\phi$ ) then the condition (70) is equivalent to

$$-\sum_{i \in \mathcal{I}} \hat{\mu}(C_i)^t \log \hat{\mu}(C_i) < \infty \quad (71)$$

**Remark 5.10.** *For infinite alphabet the function*

$$G(t) := P_G(t\phi - P_G(\phi)t) = P_G(t\phi) - P_G(\phi)t$$

is not necessarily continuous, this was first notice by Mauldin and Urbanski in [29] with their pressure. Hence, the equation  $P_G(t\phi) - P_G(\phi)t = st$  might not have a root. However, Sarig proved that if BIP holds,  $\phi$  is weakly Hölder continuous and  $P_G(t_1\phi) < \infty$ , then  $t \rightarrow P_G(t\phi)$  is real analytic for all  $t > t_1$ . See [44] and [10] for further results on the non-analyticity of  $G(t)$  in relation with the properties of recurrence of  $t\phi$ . We recall that  $G(t)$  is convex and  $G(1) = 0$  and so if  $0 < G(t_1) < \infty$ , then  $G(t_2) < G(t_1)$  for all  $t_2 > t_1$ .

From theorem 5.5 and the last two remarks follow that:

**Theorem 5.6.** *If  $\Sigma_A^{\mathcal{I}}$  satisfies the BIP property,  $\hat{\mu}$  is a  $\sigma$ -Gibbs measure for  $\phi$  with  $\sum_{n \geq 1} V_n(\phi) < \infty$ ,  $\sup \phi < \infty$ , and there exists  $0 < t_1 \leq 1$  such that  $\infty > P_G(t_1\phi) - P_G(\phi)t_1 > st_1$  and (71) holds for  $t = t_1$ , then*

$$\text{Dim}_{\hat{\mu}}(W_{\sigma}(\hat{P}, \ell_n, w)) = \sup\{t \in [t_1, 1] : P_G(t\phi) - P_G(\phi)t > st\} = \inf\{t \in (0, 1] : P_G(t\phi) - P_G(\phi)t < st\}.$$

And if moreover  $\phi$  is weakly Hölder continuous, then

$$\text{Dim}_{\hat{\mu}}(W_{\sigma}(\hat{P}, \ell_n, w)) = T \quad \text{with} \quad P_G(T\phi) - P_G(\phi)T = sT$$

**Remark 5.11.** *Theorem C of introduction follows from above theorem and remark 5.8. Notice that  $\phi(w) = -\log |g' \circ \pi(w)|$  with  $g(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$  the Gauss map.*

### 5.2.3 Non-Hölder potentials

In the previous two sections we have mainly considered potentials  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$  with some Hölder regularity property on a topologically mixing Markov chain  $\Sigma_A^{\mathcal{I}}$  verifying BI or BIP. These conditions were used to guarantee the existence of mixing local  $\sigma$ -Gibbs measures for  $t\phi$ ; however, theorems in section 5.1 only required mixing local weak  $\sigma$ -Gibbs measures.

The existence of weak Gibbs measures for non-Hölder potentials have been studied for several authors. For example, Hu considered in [22] potentials  $\phi : \Sigma_A^{\mathcal{I}} \rightarrow \mathbb{R}$ , with finite alphabet  $\mathcal{I} = \{0, 1, \dots, r^*\}$  and  $a_{0,0} = 1$ , which satisfy the Hölder condition everywhere except at the fixed point 0 and its preimages. He defined a new metric  $\tilde{d}$  on  $\Sigma_A^{\mathcal{I}}$ , the definition of this metric involves two fixed parameters  $\kappa \in (0, 1)$  and  $\gamma > 0$ , and in particular, he considered potentials  $\phi$  such that:

- (i)  $\phi$  is continuous.
- (ii) There exists  $\theta \in (0, 1]$  and  $\alpha \in [0, \theta(1 + \gamma))$  such that for all  $w \in \mathcal{B}_k$  and  $v \in \mathcal{B}_{k-1} \cup \mathcal{B}_k \cup \mathcal{B}_{k+1}$

$$|\phi(w) - \phi(v)| \leq Ck^{\alpha-1}\tilde{d}(w, v)^{\theta} \quad \text{for some constant } C > 0.$$

- (iii) There exists  $\beta > -1$  and  $K_1 > 0$  such that for all  $k \geq K_1$  if  $w \in \mathcal{B}_k$  and  $0 = (0, 0, \dots)$ , then

$$|\phi(0) - \phi(w) - \frac{\beta + 1}{k + 1}| \leq C \frac{1}{(k + 1)^{1+\delta}}$$

with  $0 < \delta \leq 1$  and  $C = C(\delta) > 0$  independent of  $k$  and  $w$ .

In properties (ii) and (iii)  $\mathcal{B}_k$  denotes the set of words  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathcal{I}}$  with  $w_0 = w_1 = \dots = w_{k-1} = 0$  and  $w_k \neq 0$ .

Due to definition of  $\tilde{d}$  and the restrictions on the values of  $\alpha$ , the condition (ii) implies a Hölder condition. Also, even though the condition (iii) implies that  $\phi$  is not a Hölder function with the usual metric, the potential  $\phi$  satisfies Hölder condition with respect to the metric  $\tilde{d}$ .

Let us assume that  $\hat{\mu}$  is a weak  $\sigma$ -Gibbs measure for the potential  $\phi$  with  $P = P_{top}(\phi) = 0$  and  $\phi$  verifying properties (i)-(iii). The measure  $\hat{\mu}$  is an ergodic equilibrium measure for  $\phi$  (see lemma 7.4 and theorem D in [22]). Moreover, for all  $0 \leq t \leq 1$  there exists an exact weak  $\sigma$ -Gibbs measure for the potential  $t\phi$  with  $P = P_{top}(t\phi)$ , see [22]. Notice that since  $P_{top}(0) > 0$  and  $P_{top}(\phi) = 0$ , follows from the convexity of  $G(t) := P_{top}(t\phi)$  that  $G(t)$  is decreasing.

Hence, we have that

$$\text{Dim}_{\hat{\mu}}(W_{\sigma}(\hat{P}, \ell_n, w)) = T \geq \frac{h_{\hat{\mu}}}{h_{\hat{\mu}} + s}$$

with  $T$  the root of the equation  $P_{top}(t\phi) = st$ .

We would like to mention that M. Yuri ([58], [59]) has established a thermodynamic formalism for *countable to one transitive Markov systems with finite range structure (FRS) satisfying a property of local exponential instability*, and he has proved the existence of equilibrium and weak  $\sigma$ -Gibbs measures for some non-Hölder potentials. See also [17]. Examples of such systems are some non-hyperbolic system which exhibit intermittent phenomena, as the Manneville-Pomeau map, we will discuss them with more detail in section 7.

## 6 Markov Transformations with BI property

Let  $\lambda$  be Lebesgue measure in  $[0, 1]$ . A map  $f : [0, 1] \rightarrow [0, 1]$  is a *Markov transformation* if there exists a finite or numerable family  $\mathcal{P}_0 = \{P_i^0\}_{i \in \mathcal{I}}$  of disjoint open intervals in  $[0, 1]$  such that

- (a)  $\lambda([0, 1] \setminus \cup_j P_j^0) = 0$ .
- (b) For each  $j$ , there exists a set  $K$  of indices such that  $f(P_j^0) = \cup_{k \in K} P_k^0 \pmod{0}$ .
- (c)  $f$  is derivable in  $\cup_j P_j^0$  and there exists  $\sigma > 0$  such that  $|f'(x)| \geq \sigma$  for all  $x \in \cup_j P_j^0$ .



- (d) There exists  $\gamma > 1$  and a non zero natural number  $n_0$  such that if  $f^m(x) \in \cup_j P_j^0$  for all  $0 \leq m \leq n_0 - 1$ , then  $|(f^{n_0})'(x)| \geq \gamma$ .
- (e) Given  $P_i, P_j \in \mathcal{P}_0$  there is  $n_0 \in \mathbb{N}$  such that  $\lambda(f^{-n}(P_i) \cap P_j) > 0$ , for all  $n \geq n_0$ .
- (f) For some  $P \in \mathcal{P}_0$ ,  $\sum_n \lambda(f^{-n}(P)) = \infty$  and  $\sum_n n \lambda([\varphi_P = n]) < \infty$ , with  $\varphi_P : P \rightarrow P$  the first return time to  $P$ , i.e.  $\varphi_P(x) = \inf\{k : f^k(x) \in P\}$ .
- (g) There exist constants  $c > 0$  and  $0 < \alpha \leq 1$  such that, for all  $x, y \in P_j^0$ ,

$$\left| \frac{f'(x)}{f'(y)} - 1 \right| \leq c|x - y|^\alpha.$$

Let us define the following collections  $\{\mathcal{P}_i\}$  of partitions:

$$\mathcal{P}_n = \bigcup_{P_i^0 \in \mathcal{P}_0} \{(f|_{P_i^0})^{-1}(P_j) : P_j \in \mathcal{P}_{n-1}, P_j \subset f(P_i^0)\} = \bigvee_{j=0}^n f^{-j}(\mathcal{P}_0);$$

from (a) and (b) we have that  $\bigcap_{n=0}^\infty \bigcup_{P \in \mathcal{P}_n} \text{cl}(P)$  has full Lebesgue measure. Let us recall some well known properties of Markov transformation (see e.g [34], [2]). From properties (d) and (g) we get the following expansiveness property: there exists  $c > 0$  and  $0 < \beta < 1$  such that for all  $x, y$  in the same interval of  $\mathcal{P}_n$

$$|x - y| \leq c\beta^n |f^n(x) - f^n(y)| \quad (72)$$

In particular we have that  $\sup_{P \in \mathcal{P}_n} \text{diam}(P) \rightarrow 0$  as  $n \rightarrow \infty$ .

Also, it is easy to check that there exists an absolute constant  $c > 0$  such that for all natural number  $n$  we have that if  $y, z$  belong to the same interval of  $\mathcal{P}_n$  then

$$\frac{(f^s)'(y)}{(f^s)'(z)} \leq c, \quad \text{for } s = 1, \dots, n+1.$$

And therefore we have the following bounded distortion property:

**Proposition 6.1.** *If  $P$  is an element of  $\mathcal{P}_n$  and  $A$  is a measurable subset of  $P$  then*

$$\frac{\lambda(A)}{\lambda(P)} \asymp \frac{\lambda(f^j(A))}{\lambda(f^j(P))}, \quad \text{for } j = 1, \dots, n+1.$$

There exists a  $f$ -invariant probability measure  $\mu$  which is absolutely continuous with respect to Lebesgue measure  $\lambda$ , usually called the ACIP measure. The measure  $\mu$  is comparable to  $\lambda$  in the initial intervals of  $\mathcal{P}_0$ , it is exact and has entropy  $h_\mu > 0$ .

Any Markov transformation is a shift modeled transformation in  $[0, 1]$  (see section 2.3). Let us denote by  $(\Sigma_A^{\mathbb{Z}}, d, \sigma)$  the symbolic representation of  $f$ . Notice that the Markov chain  $\Sigma_A^{\mathbb{Z}}$  is topologically mixing by property (e).

Let  $-\log|\widehat{f}'|$  denote the continuous extension to  $\Sigma_A^{\mathbb{Z}}$  of the potential  $-\log|f' \circ \pi|$  in  $\pi^{-1}(X_{\Pi}^{1-1})$ . Property (f) says that the potential  $-\log|\widehat{f}'|$  is positive recurrent and by property (g) and (72) follow that there exists  $c > 0$  and  $0 < \theta < 1$  such that

$$V_n(-\log|\widehat{f}'|) \leq c\theta^n \quad \text{for } n \geq 1 \quad (73)$$

Also, by property (d) we have that there exists  $\eta > 0$  such that  $-\log|(\widehat{f}^{n_0})'| \leq -\eta$  and therefore for all  $\sigma$ -invariant probability  $\widehat{\mathbf{m}}$  we have that  $\int \log|\widehat{f}'| d\widehat{\mathbf{m}} > 0$ .

We denote by  $\widehat{\mu}$  the measure in  $\Sigma_A^{\mathbb{Z}}$  corresponding to de ACIP measure  $\mu$ , i.e  $\widehat{\mu} = \mu \circ \pi$ . If  $f$  satisfies the BI property, i.e.  $\inf\{\lambda(f(P_i^0)) : P_i^0 \in \mathcal{P}_0\} > 0$ , then the measure  $\widehat{\mu}$  is a local  $\sigma$ -Gibbs measure for the potential  $-\log|\widehat{f}'|$  with  $P = P_G(-\log|\widehat{f}'|) = 0$  (see proposition 5.2).



Hereafter, we will assume that  $f$  (and therefore  $\Sigma_A^{\mathcal{I}}$ ) satisfies the BI property. We recall that if we have BIP property, then  $\hat{\mu}$  is a  $\sigma$ -Gibbs measure.

Notice that since  $\mu$  is comparable to  $\lambda$  in each initial block  $P_i^0 \in \mathcal{P}_0$ , then from lemma 2.2 we have that

$$\text{Dim}_{\Pi}(\pi(\Sigma)) = \text{Dim}_{\hat{\mu}}(\Sigma) \quad \text{for all } \Sigma \subset \Sigma_A^{\mathcal{I}}$$

## 6.1 Target-block sets for Markov transformations with BI

We say that  $x \in X_{\Pi}$  is a  $\mu$ -hitting point iff there exists a  $\hat{\mu}$ -hitting point  $w \in \Sigma_A^{\mathcal{I}}$  such that  $\pi(w) = x$ . Notice that the set of  $\mu$ -hitting points has  $\mu$ -full measure (see remark 5.1), and moreover if BIP property holds then  $\mu$  is  $\sigma$ -Gibbs and so all point in  $X_{\Pi}$  is a  $\mu$ -hitting point.

Given a  $\mu$ -hitting point  $x \in [0, 1]$  with  $\pi(w) = x$ , a block  $P \in \mathcal{P}_N$ , and a sequence  $\{\ell_n\} \subset \mathbb{N}$  such that  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$  we want to estimate the size of the set

$$W_f(P, \ell_n, x) = \{y \in P : f^k(y) \in P(\ell_k, x) \text{ for infinitely many } k\}.$$

In a similar way that in the proof of proposition 5.1 by using that  $\mu$  and  $\lambda$  are comparable in the initial blocks and proposition 6.1 we get the following upper bound for the Hausdorff dimension.

**Proposition 6.2.** *Let  $\underline{s} = \liminf_{n \rightarrow \infty} -[\log \lambda(P(\ell_n, x))]/n < \infty$ . Then*

$$\text{Dim}(W_f(P, \ell_n, x)) \leq T^+ := \inf\{t > 0 : P_G(-t \log |\hat{f}'|) < \underline{s}t\}$$

*Proof.* For each  $N \in \mathbb{N}$  we have the following covering of  $W_f(P, \ell_n, x)$

$$\bigcup_{n=N}^{\infty} \{P(n + \ell_n, y) : f^n(y) = x, \quad y \in P_i^0\}.$$

with  $P_i^0$  the 0-block such that  $P \subset P_i^0$ . Given  $\varepsilon > 0$  and  $t > 0$ , from proposition 6.1, definition of  $\underline{s}$  and since  $f^n(y) = x$ , we have for  $n$  large that

$$\frac{\lambda(P(n + \ell_n, y))}{\lambda(P(n, y))} \leq c \frac{\lambda(P(n, x))}{\lambda(P(0, x))} \leq c \frac{e^{-n(\underline{s} - \varepsilon/(2t))}}{\lambda(P(0, x))},$$

and since  $\mu$  is comparable to  $\lambda$  in  $P_i^0$

$$\lambda(P(n + \ell_n, y)) \leq c e^{-n(\underline{s} - \varepsilon/(2t))} \hat{\mu}(C(n, z))$$

with  $\pi(z) = y$  and  $c > 0$  depending on the initial block  $P_i^0$ . Hence for  $t > 0$  and  $N$  large

$$\sum_{n \geq N} \sum_{f^n(y)=x, y \in P_i^0} \text{diam}(P(n + \ell_n, y))^t \leq c \sum_{n \geq N} e^{-n(\underline{s}t - \varepsilon/2)} \sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n, z))^t.$$

But

$$\hat{\mu}(C(n, z))^t \leq c(i)^t \exp[-t \sum_{j=0}^n \log |\hat{f}' \circ \sigma^j(z)|]$$

and so for  $n$  large

$$\sum_{\sigma^n(z)=w, z_0=i} \hat{\mu}(C(n, z))^t \leq c(i)^t \sum_{\sigma^n(z)=w, z_0=i} e^{\sum_{j=0}^n (-t \log |\hat{f}' \circ \sigma^j(z)|)} \leq c e^{n[P_G(-t \log |\hat{f}'|) + \varepsilon/2]}$$

Therefore,

$$\sum_{n \geq N} \sum_{f^n(y)=x, y \in P_i^0} \text{diam}(P(n + \ell_n, x))^t \leq c \sum_{n \geq N} e^{-n[\underline{s}t - P_G(-t \log |\hat{f}'|) - \varepsilon]}.$$

For  $\varepsilon > 0$  and  $t > 0$  such that  $\underline{s}t - P_G(-t \log |\hat{f}'|) - \varepsilon > 0$  we have that

$$\sum_{n \geq N} \sum_{f^n(y)=x, y \in P_i^0} \text{diam}(P(n + \ell_n, x))^t \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty$$

□

**Remark 6.1.** A similar result holds for  $f$  a shift modeled transformation in  $[0, 1]$  (not necessarily a Markov transformation) with a  $f$ -invariant measure  $\mu$  such that  $\mu \asymp \lambda$  in the block  $P$  and  $\hat{\mu}$  a weak  $\sigma$ -invariant Gibbs measure with potential  $-\log |\hat{f}'|$  and  $P_G(-\log |\hat{f}'|) = 0$ . In this case we should change  $\lambda$  by  $\mu$  in the definition of  $\underline{s}$  (or ask for  $\mu \asymp \lambda$  in  $P(\ell_n, x)$  for all  $n$  large enough).

In this section let us assume the following condition on the center  $x$  of the target-block

$$\liminf_{n \rightarrow \infty} \frac{\lambda(P(n+1, x))}{\lambda(P(n, x))^{1+\gamma}} > 0 \quad \text{for all } \gamma > 0 \quad (74)$$

Notice that this condition holds for all  $x$  with  $\gamma = 0$  in the case of finite alphabet (follows from proposition 6.1). Moreover, in the infinite case, since  $\mu \asymp \lambda$  in  $P(0, x)$  and  $\mu$  is ergodic, then by Shannon-MacMillan-Breiman theorem follows that the set of points  $x$  such that (74) holds has full  $\lambda$ -measure. We recall that  $h_\mu > 0$ .

We also assume that

$$\bar{s} := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\lambda(P(\ell_n, x))} < \infty \quad (75)$$

**Theorem 6.1.**

$$\text{Dim}(W_f(P, \ell_n, x)) \geq T^-$$

where  $T^-$  is the supremum of all  $t > 0$  such that:

$$P_G(-t \log |\hat{f}'|) > \bar{s}t$$

and there exists a mixing local weak  $\sigma$ -Gibbs measure  $\hat{\mathbf{m}}$  for the potential  $-t \log |\hat{f}'|$  verifying (51),  $-t \log |\hat{f}'| \in L^1(\hat{\mathbf{m}})$ , and such that the collection  $\{C(0, \sigma^{p_j}(w)) : p_j \in \mathcal{I}(w)\}$  is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good for all  $\varepsilon > 0$  small enough. Here  $\pi(w) = x$  and  $w$  is a  $\hat{\mu}$ -hitting point with sequence  $\mathcal{I}(w)$ .

*Proof.* Let  $\hat{P}$  be the cylinder such that  $\pi(\hat{P}) = P$  and  $\phi = -\log |\hat{f}'|$ . We recall that  $P_G(\phi) = 0$  and  $-\int \phi d\hat{\mathbf{m}} > 0$ . Let us suppose that there exist  $t > 0$  such that  $P_G(t\phi) > t\bar{s} + \varepsilon$  for some  $\varepsilon > 0$  and there exists a mixing weak  $\sigma$ -Gibbs measure  $\hat{\mathbf{m}}$  for the potential  $t\phi$ . From theorem 5.1 we know that there exists a pattern like set  $\mathcal{Z} \subset W_\sigma(\hat{P}, \ell_n, w)$  with  $\text{Dim}_{\hat{\mu}}(\mathcal{Z}) \geq t$  and verifying also remark 5.4. Hence  $\pi(\mathcal{Z})$  is contained in  $W_f(P, \ell_n, x)$  and by lemma 2.2  $\text{Dim}_{\Pi}(\pi(\mathcal{Z})) \geq t$ . Next, we want to use remark 2.2 to conclude that  $\text{Dim}(\pi(\mathcal{Z})) = \text{Dim}_{\Pi}(\pi(\mathcal{Z}))$ . Notice that in the case of finite alphabet we have (74) for any point in  $[0, 1]$  and then we are done. Now in the infinite countable case, for all  $\gamma > 0$  we get the following by using (twice) proposition 6.1:

For  $\tilde{d}_{j+1} \leq n < d_{j+1}$

$$\begin{aligned} \frac{\text{diam}(P(n+1, \pi(z)))}{\text{diam}(P(n, \pi(z)))^{1+\gamma}} &\asymp \frac{\lambda(f^{\tilde{d}_{j+1}}(P(n+1, \pi(z))))}{\lambda(f^{\tilde{d}_{j+1}}(P(n, \pi(z))))^{1+\gamma}} \left( \frac{\lambda(f^{\tilde{d}_{j+1}}(P(n, \pi(z))))}{\lambda(P(n, \pi(z)))} \right)^\gamma \\ &\asymp \frac{\lambda(f^{\tilde{d}_{j+1}}(P(n+1, \pi(z))))}{\lambda(f^{\tilde{d}_{j+1}}(P(n, \pi(z))))^{1+\gamma}} \left( \frac{\lambda(f^{\tilde{d}_{j+1}}(P(\tilde{d}_{j+1}, \pi(z))))}{\lambda(P(\tilde{d}_{j+1}, \pi(z)))} \right)^\gamma \end{aligned}$$

Notice that  $f^{\tilde{d}_{j+1}}(P(n, \pi(z))) = P(n - \tilde{d}_{j+1}, x)$  for all  $\tilde{d}_{j+1} \leq n \leq d_{j+1}$ , since  $\sigma^{\tilde{d}_{j+1}}(C(n, z)) = C(n - \tilde{d}_{j+1}, w)$  and  $\pi(w) = x$ , (see remark 5.4). Hence, by using (74) we get that

$$\frac{\text{diam}(P(n+1, \pi(z)))}{\text{diam}(P(n, \pi(z)))^{1+\gamma}} \asymp \frac{\lambda(P(n+1 - \tilde{d}_{j+1}, x))}{\lambda(P(n - \tilde{d}_{j+1}, x))^{1+\gamma}} \left( \frac{\lambda(P(0, x))}{\lambda(P(\tilde{d}_{j+1}, \pi(z)))} \right)^\gamma \geq c_1 \quad (76)$$

for some constant  $c_1 > 0$  (depending on  $x$ ).

For  $d_j \leq n < \tilde{d}_{j+1}$ , we get in a similar way that

$$\frac{\text{diam}(P(n+1, \pi(z)))}{\text{diam}(P(n, \pi(z)))^{1+\gamma}} \asymp \frac{\lambda(f^{\tilde{d}_j}(P(n+1, \pi(z))))}{\lambda(f^{\tilde{d}_j}(P(n, \pi(z))))^{1+\gamma}} \left( \frac{\lambda(f^{\tilde{d}_j}(P(\tilde{d}_j, \pi(z))))}{\lambda(P(\tilde{d}_j, \pi(z)))} \right)^\gamma \geq c_2 \frac{\lambda(f^{\tilde{d}_j}(P(n+1, \pi(z))))}{\lambda(f^{\tilde{d}_j}(P(n, \pi(z))))^{1+\gamma}}$$

for some constant  $c_2 > 0$  (depending on  $x$ ). We recall that  $\sigma^{\tilde{d}_j}(z) \in C(\ell_{d_j}, w) \subset C(0, w)$  for all  $z \in \mathcal{Z}$  with  $\pi(w) = x$ , and therefore  $f^{\tilde{d}_j}(P(\tilde{d}_j, \pi(z))) = P(0, x)$ . Moreover, by using that  $\mu$  is comparable to  $\lambda$  in the block  $P(0, x)$  (and  $\sigma^{\tilde{d}_j}(z) \in C(0, w)$ ) we obtain from remark 5.4 that

$$\frac{\text{diam}(P(n+1, \pi(z)))}{\text{diam}(P(n, \pi(z)))^{1+\gamma}} \geq c_1 \frac{\lambda(f^{\tilde{d}_j}(P(n+1, \pi(z))))}{\lambda(f^{\tilde{d}_j}(P(n, \pi(z))))^{1+\gamma}} \asymp \frac{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n+1, z)))}{\hat{\mu}(\sigma^{\tilde{d}_j}(C(n, z)))^{1+\gamma}} \geq c \quad (77)$$

Therefore, from (76), (77) and remark 2.2 we have that  $\text{Dim}(\pi(\mathcal{Z})) = \text{Dim}_{\Pi}(\pi(\mathcal{Z})) \geq t$ .  $\square$

If the potential  $-t \log |\hat{f}'|$  is positive recurrent, then there exists the RPF measure  $\hat{\mathbf{m}}_t$  (see section 3.1) which is exact; since we have BI property and (73) holds, the measure  $\hat{\mathbf{m}}_t$  is local  $\sigma$ -Gibbs. Moreover, if there exists an increasing sequence  $\{p_j\} \subset \mathbb{N}$  such that

$$\#\{P(0, f^{p_j}(x))\} < \infty \quad (78)$$

for  $x$  the center of the target-block set (by Poincare recurrence theorem this happen  $x$   $\mu$ -a.e), then (51) holds and the collection  $\{C(0, \sigma^{p_j}(w)) : p_j \in \mathcal{I}(w)\}$  is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good for all  $\varepsilon > 0$  (since it is a finite collection). Hence we have the following corollary:

**Corollary 6.1.** *For all  $x \in X_{\Pi}$  verifying (74) and (78)*

$$\text{Dim}(W_f(P, \ell_n, x)) \geq T^- \quad \text{with}$$

$T^- := \sup\{t \in (0, 1] : \infty > P_G(-t \log |\hat{f}'|) > \overline{s}t, -t \log |\hat{f}'| \text{ positive recurrent and } -t \log |\hat{f}'| \in L^1(\hat{\mathbf{m}}_t)\}$   
Here  $\hat{\mathbf{m}}_t$  is the RPF measure for the potential  $-t \log |\hat{f}'|$ .

**Remark 6.2.** *If  $\overline{s} = 0$ , then  $\text{Dim}(W_f(P, \ell_n, x)) = 1$*

Sarig proved that if the BIP property holds, then any potential  $\phi$  with the Walters condition and  $V_1(\phi) < \infty$  and  $P_G(\phi) < \infty$  is positive recurrent. And moreover, the RPF measure is a  $\sigma$ -Gibbs measure which is exponentially continued fraction mixing. So the collection of all 0-cylinders is  $\varepsilon$ -uniformly  $\hat{\mathbf{m}}$ -good for all  $\varepsilon > 0$  (see section 3.3). Therefore

**Corollary 6.2.** *If  $\Sigma_A^{\mathcal{I}}$  satisfies the BIP property, then for all  $x \in X_{\Pi}$  verifying (74)*

$$\text{Dim}(W_f(P, \ell_n, x)) \geq T^-$$

with

$$T^- := \sup\{t \in (0, 1] : \infty > P_G(-t \log |\hat{f}'|) > \overline{s}t, \text{ and } -t \log |\hat{f}'| \in L^1(\hat{\mathbf{m}}_t)\}$$

Here  $\hat{\mathbf{m}}_t$  is the RPF measure for the potential  $-t \log |\hat{f}'|$ .

Since  $\sup -t \log |\hat{f}'| < \infty$  (due to the property (c) of the Markov transformation), by Sarig results we also know that, under BIP property, if the RPF measure  $\hat{\mathbf{m}}_t$  has finite entropy, then  $-t \log |\hat{f}'| \in L^1(\hat{\mathbf{m}}_t)$  and  $\hat{\mathbf{m}}_t$  is an equilibrium measure. Notice that if  $-\sum_{i \in \mathcal{I}} \hat{\mathbf{m}}_t(C_i) \log \hat{\mathbf{m}}_t(C_i) < \infty$ , then  $\hat{\mathbf{m}}_t$  has finite entropy, but

$$-\sum_{i \in \mathcal{I}} \hat{\mathbf{m}}_t(C_i) \log \hat{\mathbf{m}}_t(C_i) \asymp -t \sum_{i \in \mathcal{I}} \hat{\mu}(C_i)^t \log \hat{\mu}(C_i)$$

Hence, from proposition 6.2 and corollary 6.2 (see also remark 5.10) follows that:

**Corollary 6.3.** *If  $\Sigma_A^{\mathcal{I}}$  satisfies the BIP property,  $s = \lim_{n \rightarrow \infty} -[\log \lambda(P(\ell_n, x))]/n < \infty$ , and*

$$\sum_{i \in \mathcal{I}} \hat{\mu}(C_i)^{t_1} \log \hat{\mu}(C_i) < \infty \quad \text{and} \quad \infty > P_G(-t_1 \log |\hat{f}'|) > st_1 \quad \text{for some } 0 < t_1 \leq 1$$

then for all  $x \in X_{\Pi}$  verifying (74)

$$\text{Dim}(W_f(P, \ell_n, x)) = \sup\{t \geq t_1 : P_G(-t \log |\hat{f}'|) > st\} = \inf\{t > 0 : P_G(-t \log |\hat{f}'|) < st\} = T$$

with  $T$  such that  $P_G(-T \log |\hat{f}'|) = sT$

As in the proof of theorem 6.1, but using now theorem 5.2 and remark 5.6, we also get the following lower bound from the Hausdorff dimension

**Theorem 6.2.** *If  $\int \log |f'| d\mu < \infty$ , then*

$$\text{Dim}(W_f(P, \ell_n, x)) \geq \frac{\int \log |f'| d\mu}{\int \log |f'| d\mu + \bar{s}} \geq \frac{h_\mu}{h_\mu + \bar{s}}.$$

for all  $\hat{\mu}$ -hitting point  $x$  verifying (74) and such that the collection  $\{C(0, \sigma^{p_j}(w)) : p_j \in \mathcal{I}(w)\}$  is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good for all  $\varepsilon > 0$  small enough.

Recall that conditions on  $x$  in the above theorem hold  $\lambda$ -a.e., and if BIP holds then any collection of 0-cylinders is  $\varepsilon$ -uniformly  $\hat{\mu}$ -good.

Notice that by Sarig results we know that the variational principle holds for  $-\log |\hat{f}'|$  because it has summable variations and  $\sup -\log |\hat{f}'| < \infty$ . Moreover, If  $-\sum_{i \in \mathcal{I}} \mu(P_i^0) \log \mu(P_i^0) < \infty$ , then  $h_\mu = \int \log |f'| d\mu < \infty$  and  $\hat{\mu}$  is an equilibrium measure.

## 6.2 Target-ball sets for Markov transformations

Let  $x \in X_\Pi$  be a  $\mu$ -hitting point verifying (74),  $P$  a block in  $\mathcal{P}_N$ , and  $\{r_n\}$  be a decreasing sequence of positive numbers. We define

$$\widetilde{W}_f(P, r_n, x) = \{y \in P : |f^n(y) - x| \leq r_n \text{ for infinitely many } n\}.$$

Recall that from Borel-Cantelli lemma and theorem 4.1 in [15], we have the following dichotomy :

$$\begin{aligned} \text{If } \sum_n r_n < \infty & \implies \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = \infty \quad \text{for } \lambda - a.e. \ y \in [0, 1] \\ \text{If } \sum_n r_n^\alpha = \infty \text{ for some } \alpha > 1 & \implies \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = 0 \quad \text{for } \lambda - a.e. \ y \in [0, 1] \end{aligned} \quad (79)$$

If condition (74) holds for  $\gamma = 0$ , then we can take  $\alpha = 1$  in the second implication.

Notice that

$$\widetilde{W}_f(P, r_n/n, x) \subset \left\{ y \in P : \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = 0 \right\} \subset \widetilde{W}_f(P, r_n, x) \quad (80)$$

and our interest is the case  $\sum_n r_n < \infty$ . We will assume that  $u := \lim_{n \rightarrow \infty} -[\log r_n]/n < \infty$ .

For  $x \notin \partial \mathcal{P}_0$  let  $I_n := [x - r_n, x + r_n]$ ; if  $x \in \partial \mathcal{P}_0$  then  $I_n$  is half interval, i.e. an interval the length  $r_n$  and  $x$  as one of the boundary points, for example if  $x = 0$ ,  $I_n := [0, r_n]$  and if  $x = 1$ ,  $I_n := [1 - r_n, 1]$ . For all  $n$  large enough  $I_n \subset P(0, x) \cup \{x\}$ , and therefore  $f^{-n}(I_n)$  is a disjoint union of the intervals  $\text{cl}(f^{-n}(I_n) \cap P(n, y))$  with  $f^n(y) = x$ . For each  $N$  large we have the following covering of  $\widetilde{W}_f(P, r_n, x)$

$$\bigcup_{n=N}^{\infty} \{ \text{cl}(f^{-n}(I_n) \cap P(n, y)) : f^n(y) = x, \ y \in P \}.$$

But we have (see e.g. proposition 3.1 in [15])

$$\text{diam}(f^{-n}(I_n) \cap P(n, y)) = \lambda(f^{-n}(I_n) \cap P(n, y)) \leq C r_n \lambda(P(n, y))$$

Hence, proceeding as in proposition 6.2 we get that

$$\text{Dim}(\widetilde{W}_f(P, r_n, x)) \leq \inf\{t > 0 : P_G(-t \log |\hat{f}'|) < ut\}$$

On the other hand, to look for lower bounds for the dimension of this set we should notice the following: Let us define  $s_k := \text{diam}(P(p_k, x))$  with  $\mathcal{I}(x) = \{p_i\}$  the increasing sequence of the  $\mu$ -hitting point  $x$ , and for each  $s_k$  let  $n(k)$  be the greatest natural number such that  $s_k \leq r_{n(k)}$ . We

denote by  $\tilde{\mathcal{D}}$  this sequence of indexes  $\{n(k) : k \in \mathbb{N}\}$ . Since  $s_k \rightarrow 0$  as  $k \rightarrow \infty$ , and  $r_n \rightarrow 0$  as  $n \rightarrow \infty$  by hypothesis, we have that  $n(k) \rightarrow \infty$  as  $k \rightarrow \infty$ . We will write  $\tilde{\mathcal{D}} = \{\tilde{d}_i\}$  with  $\tilde{d}_i < \tilde{d}_{i+1}$  for all  $i$ . Therefore, we have that if  $\tilde{d} \in \tilde{\mathcal{D}}$  then there exists  $\ell(\tilde{d}) \in \mathcal{I}(x)$  (maybe there is more than one then we choose one) such that

$$r_{\tilde{d}+1} < \text{diam}(P(\ell(\tilde{d}), x)) \leq r_{\tilde{d}} \quad \text{and} \quad P(\ell(\tilde{d}), x) \subset I_{\tilde{d}}$$

We define the sequence  $\{\ell_n\}$  by  $\ell_n := \ell(\tilde{d}_i)$  for  $\tilde{d}_i \leq n < \tilde{d}_{i+1}$ . Since  $r_n \rightarrow 0$  as  $n \rightarrow \infty$  we have that  $\ell_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Notice that

$$V := \{y \in P : f^{\tilde{d}_i}(y) \in P(\ell_{\tilde{d}_i}, x) \text{ for infinitely many } i\} \subset \widetilde{W}_f(P, r_n, x),$$

the sequence  $\{a_n := -[\log \lambda(P(\ell_n, x))]/n\}$  is strictly decreasing for  $\tilde{d}_i \leq n < \tilde{d}_{i+1}$ , and so

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\lambda(P(\ell_n, x))} = \limsup_{i \rightarrow \infty} \frac{1}{\tilde{d}_i} \log \frac{1}{\lambda(P(\ell(\tilde{d}_i), x))} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{r_n} := u$$

We recall that the lower bounds of the Hausdorff dimension in previous section came from the  $\hat{\mu}$ -dimension of the  $(\tilde{\mathcal{J}}_j, \mathcal{J}_j)$  pattern set  $\mathcal{Z}$  constructed in theorems 5.1 and 5.2. Moreover, it is clear from the proofs of the mentioned theorems that we can construct these pattern sets with the property that  $\tilde{d}_j \in \mathcal{D}$ , and therefore  $\mathcal{Z} \subset \mathcal{V}$  and again  $\text{Dim}_{\hat{\mu}}(\mathcal{Z})$  give us the lower bounds for the Hausdorff dimension of  $\widetilde{W}_f(P, r_n, x)$ . More precisely, we have similar lower bounds for  $\text{Dim}(\widetilde{W}_f(P, r_d, x))$  as the ones in section 6.1 but with  $\bar{s} = u$ . In particular, we get the following corollary

**Corollary 6.4.** *If (74) and the BIP property holds and there exists  $0 < t_1 \leq 1$  such that*

$$-\sum_{i \in \mathcal{I}} \hat{\mu}(C_i)^{t_1} \log \hat{\mu}(C_i) < \infty \quad \text{and} \quad \infty > P_G(-t_1 \log |\hat{f}'|) > ut_1,$$

then

$$\begin{aligned} \text{Dim}(\widetilde{W}_f(P, r_n, x)) &= \sup\{t \in [t_1, 1] : P_G(-t \log |\hat{f}'|) > ut\} = \inf\{t \in (0, 1] : P_G(-t \log |\hat{f}'|) < ut\} \\ &= T \geq \frac{h_{\mu}}{h_{\mu} + u} \end{aligned}$$

with  $T$  such that  $P_G(-T \log |\hat{f}'|) = uT$

**Remark 6.3.** *Notice that it follows from (80) that under the hypothesis of corollary 6.4*

$$\text{Dim}(\widetilde{W}_f(P, r_n, x)) = \text{Dim} \left\{ y \in P : \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = 0 \right\}$$

## 6.3 Some examples

### 6.3.1 Gauss transformation and not Bernoulli modification of Gauss map

The Gauss transformation is the map  $\phi : [0, 1] \rightarrow [0, 1]$  defined by  $\phi(0) = 0$  and  $\phi(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$  for  $x \neq 0$ , where  $\lfloor x \rfloor$  denotes the integer part of  $x$ . The map  $\phi$  is a Markov transformation with partition  $\mathcal{P}_0 = \{P_i^0 := (1/(i+1), 1/i) : i \in \mathbb{N} \setminus \{0\}\}$  and such that  $\phi(P_i^0) = (0, 1)$  for all  $i$ . The symbolic representation of  $\phi$  is the full shift  $(\Sigma^{\mathbb{Z}^+}, \sigma)$ , and if  $w = (w_0, w_1, \dots) \in \Sigma^{\mathbb{Z}^+}$ , then the corresponding point in  $[0, 1]$  is the irrational point with continued fraction expansion  $[w_0, w_1, \dots]$ . Notice that  $\Sigma^{\mathbb{Z}^+}$  satisfies Bernoulli property and therefore the BIP property. The ACIP measure is the Gauss measure which is defined by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} d\lambda.$$

If  $x$  is an irrational in  $[0, 1]$  with continued fraction expansion  $[w_0, w_1, \dots]$ , then  $\lambda(P(n, x)) \asymp 1/|(\phi^{n+1})'(x)|$  and

$$\frac{1}{((w_0 + 1)(w_1 + 1) \dots (w_n + 1))^2} \leq \frac{1}{|(\phi^{n+1})'(x)|} \leq \frac{1}{(w_0 w_1 \dots w_n)^2}$$

and in order to have (74) we require for all  $\gamma > 0$

$$\liminf_{n \rightarrow \infty} \frac{(w_0 w_1 \dots w_{n-1})^\gamma}{w_n} > 0$$

for the point  $x = [w_0, w_1, \dots]$  which will be the center of the target. Now, let  $\ell_n \subset \mathbb{N}$  be a sequence such that  $\limsup_{n \rightarrow \infty} \ell_n/n < \infty$ , we ask for

$$\infty > s = s(\ell_n, x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{1}{\lambda(P(\ell_n, x))} \asymp \lim_{n \rightarrow \infty} \frac{1}{n} \log (w_0 w_1 \dots w_{\ell_n})$$

If  $\lim_{n \rightarrow \infty} \ell_n/n =: \tau$ , then by Shannon-MacMillan-Breiman theorem  $s = \tau h_\mu$  for  $\lambda$ -almost all  $x$ .

For all  $t > 1/2$  we have that

$$-\sum_{i=1}^{\infty} \mu(P_i^0)^t \log \mu(P_i^0) \asymp \sum_{i=1}^{\infty} \frac{\log i}{i^{2t}} < \infty.$$

and also for  $t > 1/2$ , we have that the function  $G(t) = P_G(-t \log |\hat{\phi}'|)$  is analytic, strictly convex and has a logarithmic singularity at  $t = 1/2$ , see e.g [37]. Hence, since  $\lim_{t \rightarrow (1/2)^+} G(t) = +\infty$  and  $G(1) = 0$ , there exists  $1/2 < T \leq 1$  such that  $G(T) = sT$

Given an irrational  $x \in [0, 1]$  and a sequence  $\{\ell_n\}$  with the above properties, we have for all block  $P \in \mathcal{P}_N$  (for some  $N$ ) the following result for recurrent patterns in continued fraction expansions (follows from corollary 6.3, and theorem 6.2)

**Theorem 6.3.** [CONTINUED FRACTIONS] *If  $\sum_n \frac{1}{(w_0 \dots w_{\ell_n})^2} < \infty \implies \lambda(W_\phi(P, \ell_n, x)) = 0$  but  $\text{Dim}(W_\phi(P, \ell_n, x)) = T \geq \frac{h_\mu}{h_\mu + s}$*

*with  $1/2 < T \leq 1$  the unique solution of  $P_G(-t \log |\hat{\phi}'|) = ts$ , and  $h_\mu = \frac{\pi^2}{6 \log 2}$ , the entropy of  $\phi$ .*

**Remark 6.4.** *If  $\sum_n \frac{1}{(w_0 \dots w_{\ell_n})^2} = \infty$ , then  $\lambda(W_\phi(P, \ell_n, x)) = 1$ , see theorem 3.1 in [15].*

**Remark 6.5.** *If  $s \geq h_\mu$  then  $h_\mu/(h_\mu + s) \leq 1/2$  and the lower bound is irrelevant. We recall that  $s = \tau h_\mu$  for  $\lambda$ -almost all  $x$ , with  $\tau := \lim_{n \rightarrow \infty} \ell_n/n$ , and so  $h_\mu/(h_\mu + s) = 1/(1 + \tau)$ .*

For  $\{r_n\}$  be a decreasing sequence of positive numbers the dichotomy (79) holds for  $\phi$ , and moreover, if  $u := \lim_{n \rightarrow \infty} -[\log r_n]/n < \infty$ , we have the following dimension result for any block  $P \in \mathcal{P}_N$

**Theorem 6.4.** *Let  $1/2 < T \leq 1$  be the unique solution of  $P_G(-t \log |\hat{\phi}'|) = tu$ , then*

$$\text{Dim} \left\{ y \in P \subset [0, 1] : \liminf_{n \rightarrow \infty} \frac{|\phi^n(y) - x|}{r_n} = 0 \right\} = T \geq \frac{h_\mu}{h_\mu + u} = \frac{\pi^2}{\pi^2 + (6 \log 2)u}$$

In particular for  $u = 0$  we have that  $T = 1$ .

We can modify the Gauss map and consider the following Markov transformation which is not Bernoulli

$$f(x) = \begin{cases} \phi(x), & \text{if } \frac{1}{2} < x \leq 1, \\ \left(1 - \frac{1}{\lfloor \frac{1}{x} \rfloor}\right) \phi(x) + \frac{1}{\lfloor \frac{1}{x} \rfloor}, & \text{if } 0 < x \leq \frac{1}{2}. \end{cases}$$

The initial partition is the same that the one for the Gauss transformation, i.e  $\mathcal{P}_0$ , but in this case  $\phi(P_1^0) = (0, 1)$  and  $\phi(P_i^0) = (1/i, 1)$  for  $i \geq 2$ . The symbolic representation of  $f$  is  $(\Sigma_A^{\mathbb{Z}^+}, \sigma)$  with transition matrix  $A = (a_{i,j})$  such that  $a_{1,j} = 1$  for all  $j \in \mathbb{Z}^+$  and for  $i > 1$  we have that  $a_{i,j} = 1$  for  $1 \leq j < i$  and  $a_{i,j} = 0$  otherwise. We would like to remark that  $f$  satisfies BIP property but it is not Bernoulli. Also, it is easy to check that if  $w = (w_0, w_1, \dots) \in \Sigma_A^{\mathbb{Z}^+}$ , then for the corresponding point  $x = \pi(w) \in [0, 1]$  we have that  $\lambda(P(n, x)) \asymp 1/|(f^{n+1})'(x)|$

$$\prod_{j=0}^n \frac{1}{g(w_j)(w_j + 1)^2} \leq \frac{1}{|(f^{n+1})'(x)|} \leq \prod_{j=0}^n \frac{1}{g(w_j)w_j^2}$$

with  $g(w_j) = 1$  if  $w_j = 1$  and  $g(w_j) = 1 - 1/w_j$  otherwise. We have the following similar result for  $f$  with  $x$  and  $u$  as in the Gauss case and  $x = \pi(w)$  such that for all  $\gamma > 0$

$$\liminf_{n \rightarrow \infty} \frac{(g(w_0)^{1/2}w_0 g(w_1)^{1/2}w_1 \dots g(w_{n-1})^{1/2}w_{n-1})^\gamma}{g(w_n)^{1/2}w_n} > 0$$

This inequality implies condition (74) which we recall holds for  $x$   $\lambda$ -a.e.

**Theorem 6.5.** *Let  $1/2 < T \leq 1$  be the unique solution of  $P_G(-t \log |\hat{f}'|) = tu$ , then*

$$\text{Dim} \left\{ y \in P \subset [0, 1] : \liminf_{n \rightarrow \infty} \frac{|f^n(y) - x|}{r_n} = 0 \right\} = T \geq \frac{\int \log |f'| d\mu}{\int \log |f'| d\mu + u}$$

with  $\mu$  the ACIP measure.

### 6.3.2 Lüroth expansion

Consider the piecewise linear Markov transformation  $f : [0, 1] \rightarrow [0, 1]$  defined by  $f(0) = 0$ ,  $f(1) = 1$  and

$$f(x) = n(n+1)x - n, \quad \text{if } x \in \left[ \frac{1}{n+1}, \frac{1}{n} \right)$$

The initial partition is  $\mathcal{P}_0 = \{P_i^0 := (1/(i+1), 1/i) : i \in \mathbb{N} \setminus \{0\}\}$ , and the symbolic representation of  $f$  is the full shift  $(\Sigma^{\mathbb{Z}^+}, \sigma)$ . If  $w = (w_0, w_1, \dots) \in \Sigma^{\mathbb{Z}^+}$ , then the corresponding point  $\pi(w) = x \in [0, 1]$  is the irrational point with *Lüroth expansion*  $[w_0, w_1, \dots]$ , i.e. (see e.g. [11])

$$x = \frac{1}{w_0 + 1} + \frac{1}{(w_0 + 1)w_0(w_1 + 1)} + \frac{1}{(w_0 + 1)w_0(w_1 + 1)w_1(w_2 + 1)} + \dots = \sum_{i=0}^{\infty} \frac{w_i}{\prod_{k=0}^n (w_k + 1)w_k}$$

Moreover,

$$\lambda(P(n, x)) = \frac{1}{(w_0 + 1)w_0(w_1 + 1)w_1 \dots (w_n + 1)w_n}$$

and we require (74) for  $x$  the center of the target. We have that

$$G(t) = P_G(-t \log |\hat{f}'|) = \log \sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right)^t,$$

and so  $G(t) = \infty$  for  $0 \leq t \leq 1/2$  and for  $t > 1/2$  we have that  $G(t)$  is real analytic, strictly decreasing and convex and has a unique zero at  $t = 1$ . Hence, we obtain for  $f$  similar results to theorems 6.3 and 6.4, we just should write  $\log \sum_{n=1}^{\infty} 1/[n(n+1)]^t$  instead of  $P_G(-t \log |\hat{\phi}'|)$ , and now  $h_\mu$  denotes the entropy of  $f$  with respect to  $\lambda$ , in fact  $h_\mu = \sum_{n=1}^{\infty} \log[n(n+1)]/n(n+1)$ .

We refers to [6] and [13] for some dimension results for other class of sets defined in terms of digit frequencies in Lüroth expansion.

### 6.3.3 Inner functions

The classical Fatou's theorem asserts that a bounded holomorphic function  $F : \mathbf{D} \rightarrow \mathbb{C}$ , from the unit disc  $\mathbf{D}$  into the complex plane  $\mathbb{C}$ , has radial limits almost everywhere. An holomorphic function  $F$  defined on  $\mathbf{D}$  and with values in  $\mathbf{D}$  is called an *inner function* if the radial limits

$$F^*(\xi) := \lim_{r \rightarrow 1^-} F(r\xi) \quad (81)$$

(which exists for almost every  $\xi$  by Fatou's theorem) have modulus 1 for almost every  $\xi \in \partial\mathbf{D}$ . It is well known that any inner function can be written as

$$F(z) = e^{i\phi} B(z) \exp \left( - \int_{\partial\mathbf{D}} \frac{\xi + z}{\xi - z} d\nu(\xi) \right)$$

where  $B(z)$  is a Blaschke product and  $\nu$  is a finite positive singular measure on  $\partial\mathbf{D}$ . We recall that given a sequence  $\{a_n\}$  in  $\mathbf{D}$  such that  $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$ , a Blaschke product  $B(z)$  is a complex function of the type

$$B(z) = z^m \prod_{a_n \neq 0} \frac{|a_n|}{a_n} \frac{z - a_n}{1 - \overline{a_n}z},$$

where  $m$  is the number of elements of the sequence  $\{a_n\}$  equal to zero. If  $F$  is inner with a fixed point  $p$  in  $\mathbf{D}$  (but  $F$  not conjugated to a rotation), Aaronson [1] and Neuwirth [35] proved, independently, that  $F^*$  is exact with respect to harmonic measure  $\omega_p$ , (see [12]). The mixing properties of inner functions are stronger, Pommerenke [36] proved that if  $F$  is an inner function with  $F(0) = 0$ , but not a rotation, then there exists a positive absolute constant  $K$  such that

$$\left| \frac{\lambda[B \cap (F^*)^{-n}(A)]}{\lambda(A)} - \lambda(B) \right| \leq K e^{-\alpha n},$$

for all  $n \in \mathbb{N}$ , for all arcs  $A, B \subset \partial\mathbf{D}$ . Here  $\alpha = \max\{1/2, |F'(0)|\}$  and  $\lambda$  denotes the normalized Lebesgue measure. If  $F$  is inner with  $F(p) = p$  and  $p \in \mathbf{D}$  the above mixing result holds for the harmonic measure  $\omega_p$ . In the terminology of [14] this imply that  $F$  is uniformly mixing at any point  $\eta$  of  $\partial\mathbf{D}$  with respect to the harmonic measure  $\omega_p$ . From Borel-Cantelli lemma and theorem 3 in [14], we have the following dichotomy for any decreasing sequence  $\{r_n\}$  of positive numbers:

$$\begin{aligned} \text{If } \sum_n r_n < \infty &\implies \liminf_{n \rightarrow \infty} \frac{d((F^*)^n(\xi), \eta)}{r_n} = \infty && \text{for } \lambda - a.e. \xi \in \partial\mathbf{D} \\ \text{If } \sum_n r_n = \infty &\implies \liminf_{n \rightarrow \infty} \frac{d((F^*)^n(\xi), \eta)}{r_n} = 0 && \text{for } \lambda - a.e. \xi \in \partial\mathbf{D} \end{aligned}$$

Here,  $d$  denotes the choral distance in  $\partial\mathbf{D}$ .

Let  $F$  be an inner function with  $F(p) = p$  and  $p \in \mathbf{D}$  and denote  $F^*(e^{2\pi i t}) = e^{2\pi i S(t)}$  and  $f(t) = S(t) \bmod 1$ . If  $f$  is a Markov transformation, then the dynamic of  $F^*$  on  $\partial\mathbf{D}$  is isomorphic to the dynamic of  $f$  and we will inherit for  $F^*$  the dimension results obtained in section 6 for Markov transformations and sequences  $\{r_n\}$  with  $u := \lim_{n \rightarrow \infty} -[\log r_n]/n < \infty$ .

*Example 1:* For  $B : \mathbf{D} \rightarrow \mathbf{D}$  a finite Blaschke product with a fixed point  $p \in \mathbf{D}$ , but not an automorphism which is conjugated to a rotation, we have

$$\text{Dim} \left\{ \xi \in \partial\mathbf{D} : \liminf_{n \rightarrow \infty} \frac{d((B^*)^n(\xi), \eta)}{r_n} = 0 \right\} = T \geq \frac{h}{h + u} \quad (82)$$

where  $T$  is the unique root of the equation  $P_{top}(-t \log |f'|) = ut$  and  $h = \int_{\partial\mathbf{D}} \log |B'(z)| d\lambda(z)$ . For  $B(z) = z^N$  we have that  $T = h/(h + u)$  and  $h = \log N$ .

The dynamic of  $B^*$  on  $\partial\mathbf{D}$  is isomorphic to the dynamic of a Markov transformation  $f$  with a finite partition  $\mathcal{P}_0$  (the number of elements of  $\mathcal{P}_0$  is the number of factors of  $B(z)$ ) and  $f(P) = [0, 1]$  for all  $P \in \mathcal{P}_0$ . The harmonic measure  $\omega_p$  is the ACIP measure.



Next we will consider other two examples where the associate transformation  $f$  is Markov with countable (but not finite) initial partition and having BIP property.

*Example 2:* Consider the infinite Blaschke product

$$B(z) = z \prod_{k=1}^{\infty} \frac{z - a_k}{1 - \overline{a_k}z}, \quad a_k = 1 - 2^{-k}.$$

Notice that  $B$  is defined and  $C^\infty$  in  $\partial\mathbf{D} \setminus \{1\}$

$$|B'(z)| = \sum_{k=0}^{\infty} \frac{1 - a_k^2}{|z - a_k|^2}, \quad \text{if } |z| = 1, \ z \neq 1,$$

and  $S'(t) = |B'(e^{2\pi it})| > C > 1$ . Moreover, it follows from Phragmén-Lindelöf theorem that the image of  $S(t)$  is  $(-\infty, \infty)$  and so for  $j \in \mathbb{Z}$  we can define the intervals  $P_j = \{t \in (0, 1) : j < S(t) < j + 1\}$ . The transformation  $f : [0, 1] \rightarrow [0, 1]$  given by  $f(t) = S(t) \pmod{1}$ ,  $f(0) = f(1) = 0$ , is a Markov transformation with partition  $\mathcal{P}_0 = \{P_j\}$  such that  $f(P_j) = (0, 1)$  and  $\lambda(P_j) \asymp 2^{-|j|}$ . The dimension result (82) holds ( $\lambda$ -a.e  $\eta \in \partial\mathbf{D}$ ) for  $B$  with  $T > 0$  the unique root of the equation  $P_G(-t \log |\hat{f}'|) = ut$ .

*Example 3:* Consider the singular inner functions

$$F(z) = e^{c \frac{z+1}{z-1}}, \quad \text{for } c > 2.$$

These inner functions have only one singularity at  $z = 1$  and its Denjoy-Wolff point  $p$  is real and it verifies  $0 < p < 1$ . We have that  $S(t) = -\frac{c}{2\pi} \cot \pi t$  for  $t \in [0, 1]$ , and  $f(t) = S(t) \pmod{1}$  is a Markov transformation with countable initial partition  $\mathcal{P}_0 = \{P_j : j \in \mathbb{Z}\}$  where  $P_j = \{t \in (0, 1) : j < S(t) < j + 1\}$  such that  $f(P_j) = (0, 1)$  and  $\lambda(P_j) = \arctan(2\pi c / (c^2 + 4\pi^2 j(j+1)))$ . We refers to [27]. The dimension result (82) holds ( $\lambda$ -a.e  $\eta \in \partial\mathbf{D}$ ) for  $F$  with  $T > 1/2$  the unique root of the equation  $P_G(-t \log |\hat{f}'|) = ut$  and  $h = \log \left( \frac{1}{1-p^2} \log \frac{1}{p^2} \right)$ .

## 7 Induced Markov transformation. Intermittent systems

We say that the map  $F : [0, 1] \rightarrow [0, 1]$  has an *induced Markov transformation* if there exist a finite or countable partition  $\mathcal{P}_0$  of the interval  $[0, 1]$  and a *return time function*  $R : \bigcup_j P_j^0 \rightarrow \mathbb{Z}^+$  which is constant on each block  $P_j^0$ , such that  $R$  is not constant almost everywhere in  $[0, 1]$  and:

- (i) The induced map  $f : [0, 1] \rightarrow [0, 1]$ , defined by  $f(y) = F^{R(y)}(y)$  (with  $R(y) = 0$  in  $[0, 1] \setminus \bigcup_j P_j^0$ ) is a Markov transformation with partition  $\mathcal{P}$
- (ii) The return time function satisfies  $\int R d\mu < \infty$  with  $\mu$  the ACIP measure associated to the Markov transformation  $f$ .

If the ACIP measure  $\mu$  is comparable to the Lebesgue measure  $\lambda$  in the whole interval  $[0, 1]$  (for example if BIP property holds), then we can write property (ii) as  $\int R d\lambda < \infty$ . Above definition includes some transformations modelled by a Young tower or intermittent interval maps as the Manneville-Pommeau transformation or the Liverani-Saussol-Vaienti transformation.

Let  $\{r_n\}$  be a decreasing sequence of positive numbers. For the Markov transformation  $f$  Borel Cantelli results state in (79) hold. Moreover, we also have (see [15]) the following result for  $F$  (instead of  $f$ ) for sequences such that  $r_n \leq C r_{2n}$  for all  $n$  with  $C$  a positive constant (as  $r_n = 1/n^{1/\alpha}$ ):

$$\text{If } \sum_n r_n^\alpha = \infty \text{ for some } \alpha > 1 \implies \liminf_{n \rightarrow \infty} \frac{|F^n(y) - x|}{r_n} = 0 \quad \text{for } \lambda - a.e. \ y \in [0, 1]$$

If  $\lambda(P(n, x)) \asymp \lambda(P(n+1, x))$  for all  $n$  (with constants depending on  $x$ ), then the last implication also holds for  $\alpha = 1$ .

If we denote by  $R_n(y) = \sum_{k=0}^{n-1} R(f^k(y))$ , then  $f^n(y) = F^{R_n(y)}(y)$ . We can state all the previous dimension results on target sets for the Markov transformation  $f$  in terms of the map  $F$ . More precisely, we have dimension results for the set

$$\{y \in [0, 1] : |F^{R_n(y)}(y) - x| \leq r_n \text{ for infinitely many } n\}$$

However, if we want to get dimension results for the set

$$\{y \in [0, 1] : |F^n(y) - x| \leq r_n \text{ for infinitely many } n\},$$

then we will need to study the existence of weak  $\sigma$ -Gibbs measures for the potentials  $-t \log |\widehat{F}'|$  instead of  $-t \log |\widehat{f}'|$ , and also the mixing properties of these measures. For some intermittent systems the existence of equilibrium measure which are weak  $\sigma$ -Gibbs (in the symbolic context) for these potentials is well know. So, we would be able to use our results for Markov shift and weak Gibbs measures in these settings. For the existence of weak Gibbs measure for intermittent system see [38], [59], [25], [23], [22].

## 7.1 The Manneville-Pomeau model for intermittency

Let us consider the family of transformations  $F : [0, 1] \rightarrow [0, 1]$  with

$$F(x) = x + x^{1+\alpha} \pmod{1} \quad \text{with} \quad 0 < \alpha < 1$$

These maps are uniformly expanding out of all neighborhood of the fixed point 0. We know (see [37]) that for the countable partition  $\{P_j^0\}$

$$P_0^0 = (a_0, 1) \quad P_j^0 = (a_j, a_{j-1}) \quad \text{with} \quad a_0 + a_0^{1+\alpha} = 1 \quad \text{and} \quad F(a_{j+1}) = a_j$$

and the return time function defined by  $R(y) = j$  iff  $y \in P_{j+1}^0 = (a_{j+1}, a_j)$ , the induced map  $f(y) = F^{R(y)}(y)$  satisfies conditions (i) and (ii). Notice that  $f$  has BIP property, since  $f(P_j^0) = (0, 1)$  for all  $j$ , and the condition (ii) is  $\sum_j j \lambda(P_j^0) < \infty$  but

$$\lambda(P_j^0) = a_{j-1} - a_j = a_j^{1+\alpha} \asymp \frac{1}{(\alpha(j+1))^{1+1/\alpha}}$$

If we denote, as usual, the symbolic representation of  $f'$  as  $\widehat{f}'$ , then it is easy to check that  $G(t) = P_G(-t \log |\widehat{f}'|)$  for  $t \in (\alpha, 1]$  is continuous, convex,  $G(1) = 0$  and  $\lim_{t \rightarrow \alpha^+} G(t) = \infty$ . Also, for  $t \in (\alpha, 1]$

$$-\sum_j \lambda(P_j^0)^t \log \lambda(P_j^0) \asymp \sum_j \frac{1}{j^{t(1+1/\alpha)}} \log(\alpha j) < \infty$$

We ask  $x \in X_\Pi$  be a point such that (74) holds, and also we require the decreasing sequence  $\{r_n\}$  verifies  $\sum_n r_n < \infty$  and  $0 \leq u := \lim_{n \rightarrow \infty} -(\log r_n)/n < \infty$ . For any block  $P \in \mathcal{P}_N$  for some  $N$ , the set

$$\widetilde{W}_f(P, r_n, x) = \left\{ y \in P : \liminf_{n \rightarrow \infty} \frac{|F^{R_n(y)}(y) - x|}{r_n} = 0 \right\}$$

with  $f(y) = F^{R(y)}(y)$  verifies the following:

**Theorem 7.1.** *Let  $\alpha < T \leq 1$  be the unique solution of  $P_G(-t \log |\widehat{f}'|) = tu$ , then*

$$\lambda(\widetilde{W}_f(P, r_n, x)) = 0 \quad \text{and} \quad \text{Dim}(\widetilde{W}_f(P, r_n, x)) = T$$

Moreover, for  $u > (1 + \frac{1}{\alpha})h_\mu$

$$T \geq h_\mu / (h_\mu + u)$$

Here  $h_\mu$  is the entropy of the ACIP measure of  $f(y) = F^{R(y)}(y)$ .

If  $\sum_n r_n = \infty$  we have from Borel-Cantelli lemma that  $\lambda(\widetilde{W}_f(P, r_n, x)) = 1$ .

Next, we will look for a Hausdorff dimension result for  $F$  instead of  $f(y) = F^{R(y)}(y)$ . Let us consider now the initial partition  $\mathcal{P}^0 = \{(0, a_0), (a_0, 1)\}$ . Notice that the full shift  $(\Sigma^{\{0,1\}}, \sigma)$  gives a symbolic representation of  $F$  and we have that  $P_{top}(-\log |F'|) = 0$ . It is known (see [36]) that there exists an ACIP measure  $\mu$ . The density function  $h(x)$  of  $\mu$  verifies  $h(x) \asymp x^{-\alpha}$  for all  $x > 0$ , and there exist constants  $C > C' > 0$  and  $n_0 > 0$  such that for any measurable set  $E \subset [0, 1]$  and any  $m$ -block  $P \in \mathcal{P}_m = \bigvee_{j=0}^m f^{-j}(\mathcal{P}^0)$

$$|\mu(F^{-k}(E) \cap P) - \mu(E)\mu(P)| \leq C \frac{m^{\frac{1}{\alpha}-1}}{(k-m-n_0)^{\frac{1}{\alpha}-1}} \mu(E)\mu(P) \quad \text{for all } k > n_0 + m$$

and if  $P = P(m, 0)$  with  $m \geq n_0$

$$|\mu(F^{-k}(E) \cap P) - \mu(E)\mu(P)| \geq C \frac{m^{\frac{1}{\alpha}-1}}{k^{\frac{1}{\alpha}-1}} \mu(E)\mu(P) \quad \text{for all } k > m$$

We refers to [21] and [22], an also [57]. Notice that in particular,

$$\mu(F^{-k}(E) \cap P) \leq (1 + C \frac{m^{\frac{1}{\alpha}-1}}{(k-m-n_0)^{\frac{1}{\alpha}-1}}) \mu(E)\mu(P) \quad \text{for all } k > n_0 + m$$

and so for any  $k$ -block  $P(k, z)$

$$\mu(F^{-k}(E) \cap P(k, z)) \leq \mu(F^{-k}(E) \cap P(k - n_0 - 1, z)) \leq C' \mu(E)\mu(P(k - n_0 - 1, z)) \quad (83)$$

with  $C' = 1 + C(k - n_0 - 1)^{\frac{1}{\alpha}-1}$ .

Moreover, the measure  $\widehat{\mu}$  in  $\Sigma^{\{0,1\}}$  is a weak  $\sigma$ -Gibbs measure for the potential  $\phi := -\log |\widehat{F}'|$  and an equilibrium measure (see [22]). In fact, for all  $0 \leq t \leq 1$  there exists an exact weak  $\sigma$ -Gibbs measure  $\widehat{\mathbf{m}}_t$  for the potential  $t\phi$  with  $P = P_{top}(t\phi)$  which is an equilibrium measure (see theorem F in [22]). Notice that, since  $\widehat{\mu}$  is weak Gibbs it follows from (83) that given  $\varepsilon > 0$  for all  $k$  large enough (depending on  $\varepsilon$ ,  $\alpha$  and  $n_0$ )

$$\mu(F^{-k}(E) \cap P(k, z)) \leq \mu(F^{-k}(E) \cap P(k - n_0 - 1, z)) \leq e^{\varepsilon k} \mu(E) \widehat{\mu}(C(k, w)) \quad (84)$$

with  $w \in \Sigma^{\{0,1\}}$  such that  $\pi(C(k, w)) = \text{cl}(P(k, z))$

Let  $P$  denote a  $N$ -block of  $\mathcal{P}_N$  such that  $0 \notin P$  (this condition allows to use that  $\lambda \asymp \mu$  in  $P$ ),  $x$  be (any) point in  $[0, 1]$ , and let us suppose again that  $\sum_n r_n < \infty$  and

$$\widetilde{u} := \lim_{n \rightarrow \infty} -(\log \mu(I_n))/n < \infty \quad (85)$$

with  $I_n := [x - r_n, x + r_n]$  if  $x \neq 0, a_0, 1$ , and  $I_n := [0, r_n]$  if  $x = 0$ ,  $I_n = [a_0 - r_n, a_0]$  if  $x = a_0$  and  $I_n := [1 - r_n, 1]$  if  $x = 1$ . Notice that if  $u := \lim_{n \rightarrow \infty} -(\log r_n)/n$ , then since the density function of  $\mu$  verifies  $h(x) \asymp x^{-\alpha}$  we have that

$$\widetilde{u} = \begin{cases} u, & \text{if } x \neq 0 \\ (1 - \alpha)u, & \text{if } x = 0. \end{cases}$$

Then we have the following result:

**Theorem 7.2.** *Let  $0 \leq T \leq 1$  be the unique solution of  $P_{top}(-t \log |F'|) = t\widetilde{u}$ , then*

$$\text{Dim} \left\{ y \in P : \liminf_{n \rightarrow \infty} \frac{|F^n(y) - x|}{r_n} = 0 \right\} = T \geq h_\mu / (h_\mu + \widetilde{u})$$

with  $h_\mu$  is the entropy of the ACIP measure of  $F$ .

Notice that the Hausdorff dimension is bigger for  $x = 0$ .

We will need the following distortion estimates whose proof is similar to the one of proposition 2.3 in [22]. Recall  $F(a_0) = a_0 + a_0^{1+\alpha} = 1$  and  $F(a_{j+1}) = a_j$ ; we define  $a_{-1} = 1$ .

**Lemma 7.1.** *There exist  $C_1, C_2 > 0$  and  $j_0 \geq 0$  such that if*

$$z, y \in (a_j, a_{j-1}] \quad \text{and} \quad F(z), F(y) \in (a_k, a_{k-1}] \quad \text{for some } j, k \geq 0$$

then

$$\Delta(z, y) \frac{F'(z)}{F'(y)} \leq \Delta(F(z), F(y))$$

with

$$\Delta(u, v) = \begin{cases} 1 + C_1 \frac{|u - v|}{u}, & \text{if } u, v \in (a_j, a_{j-1}] \text{ with } j > j_0 \\ 1 + C_2 |u - v|, & \text{if } u, v \in (a_j, a_{j-1}] \text{ with } j \leq j_0 \end{cases}$$

Notice that since  $F((a_{j+1}, a_j]) = (a_j, a_{j-1}]$  for  $j \geq 0$  we have the following :  
Let  $\pi(z_0, z_1, \dots) = z$  and  $\pi(y_0, y_1, \dots) = y$  (with  $\pi : \Sigma^{\{0,1\}} \rightarrow [0, 1]$  the natural projection) and  $\mathcal{Q} = \{(a_j, a_{j-1}] : j \geq 0\}$

- (a) If  $z_m = 1$  and  $y_n = z_n$  for  $0 \leq n \leq m$ , then  $F^n(z), F^n(y)$  belong to the same element of  $\mathcal{Q}$
- (b) If  $y_n = 0$  for  $0 \leq n \leq m-1$ , then  $y \in [0, a_{m-1}]$  and  $F^n(y) \in [F^{n-1}(y), a_{m-1-n}]$ . Moreover, if  $y_m = 1$ , then  $y \in [a_m, a_{m-1}]$

From this observation and lemma 7.1 we have that:

**Lemma 7.2.** (i) *If  $y \in P(m, z)$  and  $z_m = 1$ , then for all  $0 < s \leq m$*

$$(F^s)'(z) \asymp (F^s)'(y)$$

- (ii) *If  $y \in P(m, z) \neq P(m, 0)$  with  $z_r = 1$  and  $z_i = 0$  for  $r+1 \leq i \leq m$ , then  $(F^r)'(y) \asymp (F^r)'(z)$  and for all  $r < s \leq m$*

$$(F^s)'(y) \asymp (F^r)'(z)(F^{s-r-1})'(a_{k(y)-r})$$

with  $k(y) = \min\{k : k > m \text{ and } y_k = 1\}$ . If  $y \in P(m, 0)$ , then  $(F^s)'(y) \asymp (F^s)'(a_{k(y)})$

*Proof.* Part (i) follows from lemma 7.1 (as in proposition 2.3 (ii) in [22]) and property (a) since

$$\frac{(F^s)'(z)}{(F^s)'(y)} = \prod_{j=0}^{s-1} \frac{(F)'(F^j(z))}{(F)'(F^j(y))} \leq \frac{\Delta(F^s(z), F^s(y))}{\Delta(z, y)}$$

We recall that  $F^n(z), F^n(y)$  (for  $0 \leq n \leq m$ ) belongs to the same element of  $\mathcal{Q}$ , say  $(a_j, a_{j-1}]$ , and so

$$\Delta(F^s(z), F^s(y)) - 1 \leq \max\{C_1, C_2\} \frac{a_{j-1} - a_j}{a_j} = \max\{C_1, C_2\} a_j^\alpha \leq cte$$

From part (ii) notice that  $y_r = y_k = 1$  and  $y_i = 0$  for  $r < i < k$ , then from (i) we have that  $(F^r)'(y) \asymp (F^r)'(z)$  and  $(F^{s-r-1})'(F^{r+1}(y)) \asymp (F^{s-r-1})'(a_{k-r})$ . Recall  $(F)'(F^r(y)) \asymp 1$ .  $\square$

**Corollary 7.1.** *Let  $1 \leq s \leq n$ .*

- (i) *If  $z_i = 1$  for some  $s \leq i \leq n$ , then*

$$\frac{\lambda(F^s(P(n+1, z)))}{\lambda(F^s(P(n, z)))} \asymp \frac{\lambda(P(n+1, z))}{\lambda(P(n, z))}$$

(ii) If  $z_r = 1$  for some  $0 \leq r < s$  and  $z_i = 0$  for  $r+1 \leq i \leq n$ , then for all  $r < s \leq n$

$$\frac{\lambda(P(n+1, z))}{\lambda(P(n, z))} \asymp \left( \frac{n-r}{n+1-r} \right)^{1/\alpha} \geq c \frac{\lambda(F^s(P(n+1, z)))}{\lambda(F^s(P(n, z)))} \quad \text{if } z_{n+1} = 0$$

and

$$\frac{\lambda(P(n+1, z))}{\lambda(P(n, z))} \asymp \frac{(n-r)^{1/\alpha}}{(n+1-r)^{1+1/\alpha}} \geq c \frac{\lambda(F^s(P(n+1, z)))}{\lambda(F^s(P(n, z)))} \quad \text{if } z_{n+1} = 1$$

for some positive constant  $c$ .

(iii) If  $z_i = 0$  for  $0 \leq i \leq n$ , then for all  $0 \leq s \leq n$  the estimates in (ii) holds with  $r = -1$ .

**Remark 7.1.** Notice that in particular we have that

$$\inf_x \liminf_{n \rightarrow \infty} \lambda(P(n+1, x)) / \lambda(P(n, x)) > 0.$$

*Proof.* From lemma 7.2 (i) we have  $\lambda(F^s(P(k, z))) \asymp (F^s)'(z) \lambda(P(k, z))$  (for  $k = n, n+1$ ), and we get part (i). From lemma 7.2 (ii) we know that if  $y \in P(n, z)$  then  $(F^{r+1})'(y) \asymp (F^{r+1})'(z)$ . Therefore if  $z_{n+1} = 0$

$$\begin{aligned} \frac{\lambda(P(n+1, z))}{\lambda(P(n, z))} &= \frac{\lambda(P(n+1, z))(F^{r+1})'(z)}{\lambda(P(n, z))(F^{r+1})'(z)} \asymp \frac{\lambda(F^{r+1}(P(n+1, z)))}{\lambda(F^{r+1}(P(n, z)))} = \frac{\lambda(P(n-r, 0))}{\lambda(P(n-r-1, 0))} = \frac{a_{n-r}}{a_{n-r-1}} \\ &\asymp \left( \frac{n-r}{n+1-r} \right)^{1/\alpha} \geq \left( \frac{n-(s-1)}{n+1-(s-1)} \right)^{1/\alpha} \asymp \frac{\lambda(P(n+1-s, 0))}{\lambda(P(n-s, 0))} = \frac{\lambda(F^s(P(n+1, z)))}{\lambda(F^s(P(n, z)))} \end{aligned}$$

In a similar way, if  $z_{n+1} = 1$ , then  $\lambda(F^{r+1}(P(n+1, z))) = \lambda(P(n-r-1, 0)) - \lambda(P(n-r, 0)) = a_{n-r-1} - a_{n-r} = a_{n-r}^{1+\alpha}$  and we get

$$\frac{\lambda(P(n+1, z))}{\lambda(P(n, z))} \asymp \frac{a_{n-r}^{1+\alpha}}{a_{n-r-1}} \asymp \frac{(n-r)^{1/\alpha}}{(n+1-r)^{1+1/\alpha}}.$$

By using that  $\lambda(F^s(P(n+1, z))) = \lambda(P(n-s, 0)) - \lambda(P(n+1-s, 0)) = a_{n-s} - a_{n+1-s} = a_{n+1-s}^{1+\alpha}$  we get (ii). Finally (iii) follows from taking  $r = -1$  in the proof of (ii)  $\square$

*Proof. of theorem 7.2.* Since (80) holds for  $F$ , an upper bound for the dimension follows from the upper bound for  $\text{Dim}(\widetilde{W}_F(P, r_n, x))$  with

$$\widetilde{W}_F(P, r_n, x) = \{y \in P : |F^n(y) - x| \leq r_n \text{ for infinitely many } n\}$$

For all  $N$  large enough we have the following covering of  $\widetilde{W}_F(P, r_n, x)$

$$\bigcup_{n=N}^{\infty} \{\text{cl}(f^{-n}(I_n) \cap P(n, y)) : f^n(y) = x, \quad y \in P\}.$$

with  $I_n := [x - r_n, x + r_n]$  if  $x \neq 0, a_0, 1$ , and  $I_n := [0, r_n]$  if  $x = 0$ ,  $I_n = [a_0 - r_n, a_0]$  if  $x = a_0$  and  $I_n := [1 - r_n, 1]$  if  $x = 1$ . Notice that for  $n$  large enough  $I_n \subset P(0, x) \cup \{x\}$ .

From (84) and since  $\lambda \asymp \mu$  in  $P$ , we have that the diameter of each interval  $F^{-n}(I_n) \cap P(n, y)$  verifies

$$\text{diam}(F^{-n}(I_n) \cap P(n, y)) = \lambda(F^{-n}(I_n) \cap P(n, y)) \leq e^{\varepsilon n} \mu(I_n) \widehat{\mu}(C(n, w))$$

with  $w \in \Sigma^{\{0,1\}}$  such that  $\pi(C(n, w)) = \text{cl}(P(n, y))$  From this inequality and by definition of  $\widetilde{u}$ , see (85), we can proceed as in proposition 6.2 (see remark 6.1) and to get

$$\text{Dim}(\widetilde{W}_F(P, r_n, x)) \leq \inf\{t > 0 : P_{top}(-t \log |F'|) < \widetilde{u} t\}$$

Again, since (80) holds for  $F$  we can get a lower bound for the dimension of our set by obtaining a lower bound for  $\text{Dim}(\widetilde{W}_F(P, r_n/n, x))$ . Notice that the value of  $\tilde{u}$  for the sequence  $\{r_n/n\}$  coincides with the corresponding value for the sequence  $\{r_n\}$ . We apply the same arguments that in the case of target-ball sets for Markov transformations (see section 6.2), we define (in the same way) a target-block set  $W_F(P, \ell_n, x) \subset \widetilde{W}_F(P, r_n/n, x)$ . As in the proof of theorems 6.1 and 6.2, we use the symbolic patterns sets  $\mathcal{Z}$  given by theorems 5.1 and 5.2. Notice that we have remark 7.1 and so, from proposition 2.1, we have equality between the Hausdorff dimension and the grid dimension of  $\pi(\mathcal{Z})$   $\square$

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